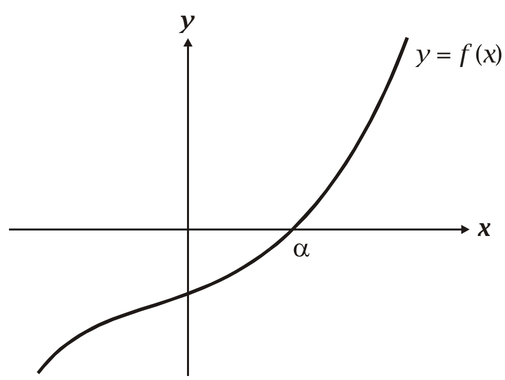
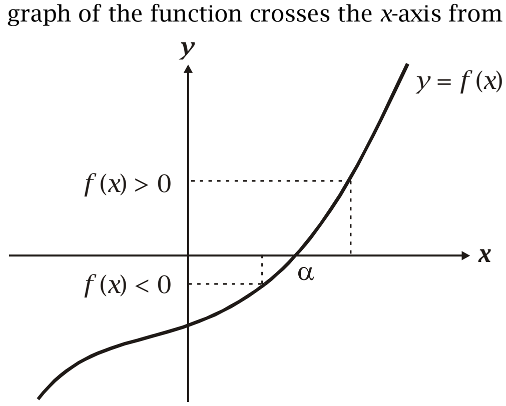
Sketching graphs to illustrate solutions to roots of equations

The method of trial and improvement

A root of an equation is a value that makes that equation zero. If $y = f(x)$ is a function, then a *root* of the equation $f(x) = 0$ is a value $x = \alpha$ such that $f(\alpha) = 0$.

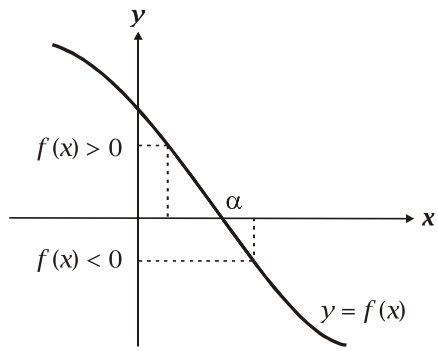


Around the root α the value of the function $y = f(x)$ changes sign. For example, suppose the graph of the function crosses the *x*-axis from below, as *x* increases.

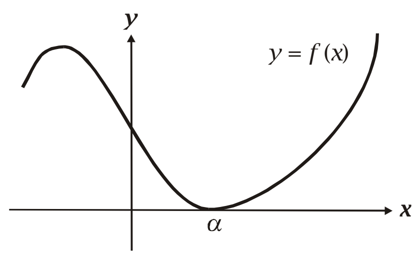


In this case when $x < \alpha$ then $f(x) < 0$ and when $x > \alpha$ then $f(x) > 0$.

The situation is reversed when the graph is decreasing as it crosses the *x*-axis.



There is also the possibility that the graph will just touch the *x*-axis at a minimum or maximum. For example



The graph indicates whether the function cuts the *x*-axis at the root from above or below, and hence whether there is a sign change in $y = f(x)$ around the root. If there is a sign change we can use the method of "trial and improvement" to find the numerical value of the root to a required degree of accuracy. The idea of this method is that we progressively narrow the size of interval in which the root must lie using the fact that at the ends of any such interval must have opposite signs – one end positive, the other negative.

Example (1)

The positive root to the equation $4x^2 - 3x - 2 = 0$ lies in the interval $[1,2]$. Find the value of this root to 2 decimal places.

Solution

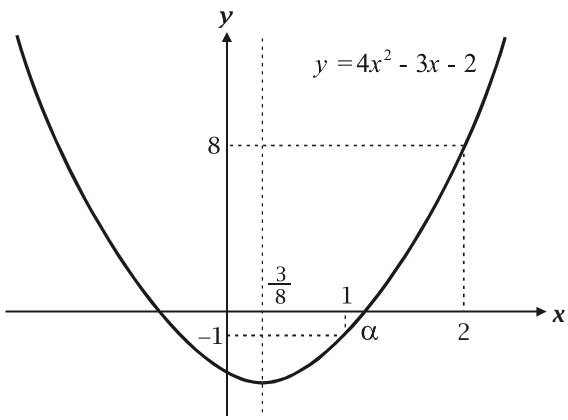
Firstly, note that $f(x) = 4x^2 - 3x - 2$ is a quadratic polynomial, and can be sketched by means of completing the square, as follows

$$\begin{aligned} f(x) &= 4x^2 - 3x - 2 \\ &= 4\left(x^2 - \tfrac{3}{4}x\right) - 2 \\ &= 4\left(x^2 - \tfrac{3}{4}x + \left(\tfrac{3}{8}\right)^2 - \left(\tfrac{3}{8}\right)^2\right) - 2 \\ &= 4\left(x^2 - \tfrac{3}{4}x + \left(\tfrac{3}{8}\right)^2\right) - \tfrac{9}{16} - \tfrac{32}{16} \\ &= 4\left(x - \tfrac{3}{8}\right)^2 - \tfrac{41}{16} \end{aligned}$$

Also note

$f(1) = -1$
$f(2) = 8$

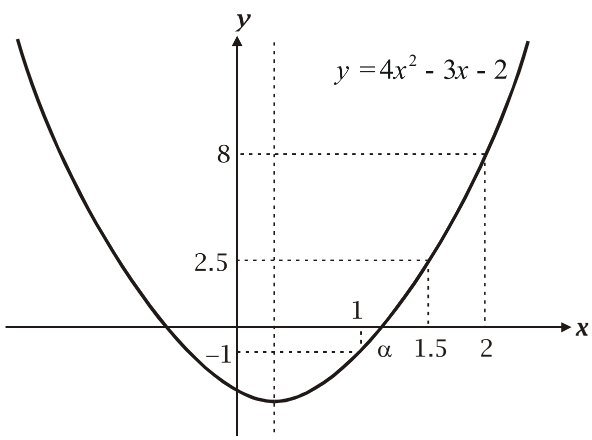
So the graph of $f(x) = 4x^2 - 3x - 2$ is



Since there is a sign change between $x = 1$ and $x = 2$ the root α must lie in this interval. We endeavour to narrow this interval by testing the mid point.

$f(1.5) = 2.5$

This is positive so there is no sign change in the interval $[1.5, 2]$. However, there is a sign change in the interval $[1, 1.5]$, so α lies in this interval.



We "home in" on the required interval as follows

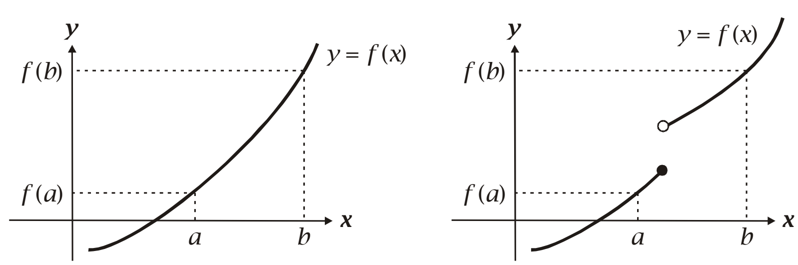
$f(1) = -1$	
$f(2) = 8$	$1 < \alpha < 2$
$f(1.5) = 2.5$	$1 < \alpha < 1.5$
$f(1.2) = 0.16$	$1 < \alpha < 1.2$
$f(1.1) = -0.46$	$1.1 < \alpha < 1.2$
$f(1.15) = -0.16$	$1.15 < \alpha < 1.20$

$f(1.17) = -0.0344 \qquad 1.17 < \alpha < 1.20$
$f(1.18) = 0.0296 \qquad 1.17 < \alpha < 1.18$
$f(1.175) = -0.0025 \qquad 1.175 < \alpha < 1.18$
Therefore, $\alpha = 1.18$ (2 d.p.)

In this question note how we have to test a further point x = 1.175 in order to determine whether the root, α, lies closer to 1.18 or to 1.17 to 2 decimal places. This question illustrates the method of *trial and improvement* for finding a root to an equation. In this method we successively home in on the root using the fact that the root must lie in an interval $[a, b]$ where $f(a) < 0$ and $f(b) > 0$ or vice-versa. This method is the most basic method for finding a root to an equation. It is laborious and slow. Later you will learn other methods that are faster, but have the disadvantage that they do not always "work".

The intermediate value theorem

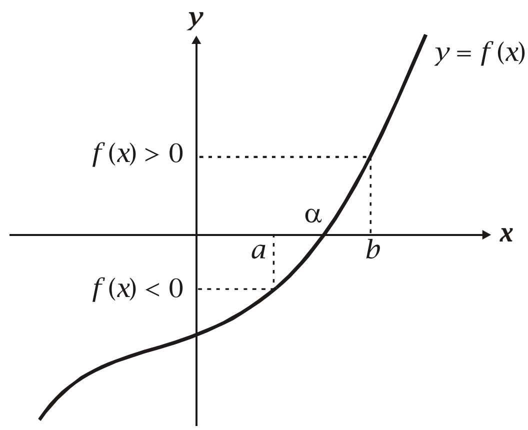
The method of trial and improvement rests on a theorem about *continuous* functions called the *intermediate value theorem*. Intuitively, a function is *continuous* if its graph consists of one unbroken curve. If you were drawing the graph with a pencil you would draw the graph as one curve or line *without lifting your pencil*. This is the intuitive notion of a continuous graph. When we say that the root must lie in an interval $[a, b]$ where $f(a) < 0$ and $f(b) > 0$ or vice-versa we are assuming the *intermediate value theorem*[1]. This theorem states that in an interval $[a, b]$ a continuous function takes every value between $f(a)$ and $f(b)$



Continuous *Discontinuous*

We do not intend to prove this theorem at this stage, which properly belongs to a more advanced chapter. However, at this level a question that refers to the intermediate value theorem is asking you to appreciate that the root of an equation, which is a value where $f(x) = 0$, must lie in an interval $[a, b]$ where $f(a) < 0$ and $f(b) > 0$ or vice-versa provided the function f is continuous.

[1] Also called *Bolzano's Theorem* after the Czech mathematician Bernhard Bolzano (1781 – 1848)



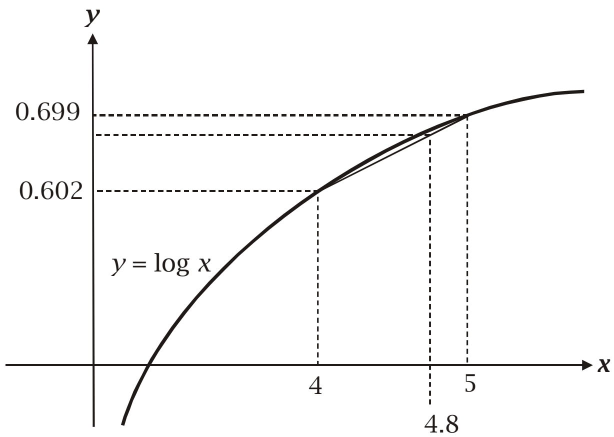
In the above diagram it is by the intermediate value theorem that the root α must lie in the interval $[a, b]$.

Linear interpolation

The method of trial and error is an example of a *numerical method* for finding a root. It contrasts with exact methods such as the algebraic method of finding the root of a quadratic by the technique of completing the square. Numerical methods depend on a starting value, so it often helps if the starting value is as close to the real root as possible. Of course, we cannot really know for certain if the starting value is close to the true value of the root, but if we believe that the function $f(x)$ is continuously increasing or decreasing, the shape of the curve may be quite close to that of a line, so a technique known as *linear interpolation* may be used to get a good approximation to the root in the first instance. We will explain this by means of an example.

Example (2)
Given that $\log 4 = 0.602,\ \log 5 = 0.699$, use linear interpolation to find an approximation to $\log 4.8$. (Note, this question, these are logs to the base 10.)

Solution
The function $f(x) = \log_{10} x$ is a continuously increasing function.
We are given $\log 4 = 0.602,\ \log 5 = 0.699$ which give us points on a graph $(4, 0.602),\ (5, 0.699)$. To estimate $\log 4.8$ we assume a linear relationship between these two points; in other words, we join them by a line. When we have done so we shall use ratios to find the approximate solution to $\log 4.8$.



$$\ln(4.8) \approx \ln(4) + 0.8 \times (0.699 - 0.602)$$
$$= 0.602 + 0.0776$$
$$= 0.6796$$

Sketching curves to illustrate solutions to equations

The graphs of two functions, $y = f(x)$ and $y = g(x)$, meet when

$f(x) = g(x)$

So sketching the graphs of the two functions will provide information about the number of solutions to the equation $f(x) = g(x)$, and where these solutions lie. Once this information has been obtained, the method of trial and improvement can be used to find the approximate value of x that makes $f(x) = g(x)$ true. The application of these ideas is best illustrated by example.

Example (3)

Sketch the curve, C, whose equation is

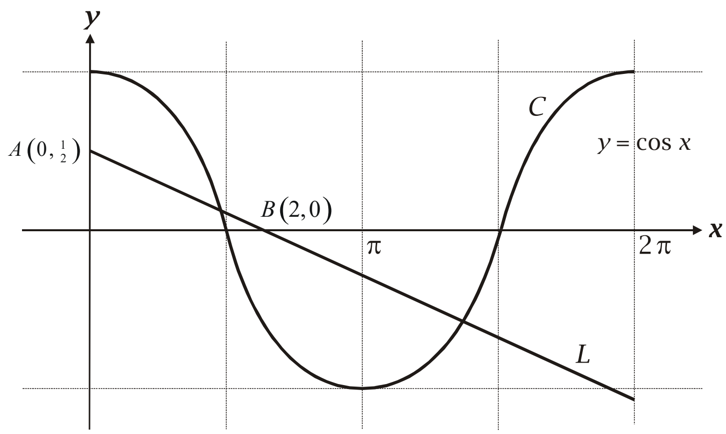
$y = \cos x$ for $0 \le x \le 2\pi$

Mark on this diagram the points

$A\left(0, \frac{1}{2}\right)$ $B(2,0)$

Find the equation of the line, L, which passes through A and B, expressing your answer in the form $y = mx + c$. Write down an equation satisfied by the x-coordinates of the intersection of L and C. State the number of solutions in the interval $0 \le x \le 2\pi$ to this equation. Find to 1 decimal place the smallest positive value of x that satisfies this equation.

Solution



To find the equation of the line, *L*, passing through *A* and *B*, let the gradient be *m*. Then $m = \frac{\Delta y}{\Delta x} = \frac{-\frac{1}{2}}{2} = -\frac{1}{4}$ and the intercept is $c = \frac{1}{2}$. Therefore, on substitution into $y = mx + c$ we obtain

$$y = -\frac{1}{4}x + \frac{1}{2}$$

The equation satisfied by the *x*-coordinates of the intersection of *L* and *C* is

$$\cos x = -\frac{1}{4}x + \frac{1}{2}$$

From the sketch we can see that this has two roots in the interval $0 \le x \le 2\pi$. To find the smallest positive value of *x* that satisfies this equation, we use the method of trial and improvement to look for a sign change. Let

$$f(x) = \cos x + \frac{1}{4}x - \frac{1}{2}$$

then

$f(1.5) = -0.0542..$

$f(1.4) = 0.019967..$ $\qquad 1.4 < \alpha < 1.5$

$f(1.45) = -0.01699...$ $\qquad 1.4 < \alpha < 1.45$

and therefore $\alpha = 1.4 \;(1 \text{ d.p.})$.