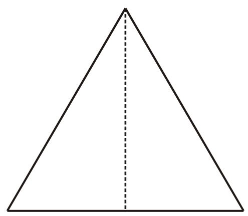
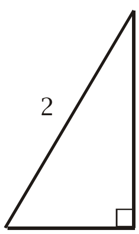
Special Triangles

Exact values of trigonometric functions

It is possible to find the exact values of the sine, cosine and tangent of certain important and frequently used angles. These ratios are given by two "special" triangles. One of these is half an equilateral triangle.



Let one of the sides of this triangle by 2 units in length.

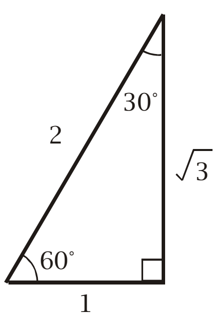


Example (1)

(*a*) In the triangle above determine by means of Pythagoras's Theorem the lengths of the other two sides.

Solution

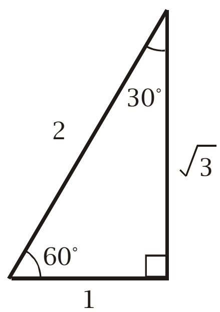
Let the length of one side to be 2 units in length, then the base is 1 unit and the altitude (height) is $\sqrt{3}$ by Pythagoras.



Example (1) continued

(*b*) Deduce from the triangle drawn in example (1) the *exact* values (using surds if necessary) of $\cos 60$, $\cos 30$, $\sin 30$, $\sin 60$, $\tan 60$ and $\tan 30$.

Solution



Substituting into the familiar trigonometric ratios.

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

$$\cos 30 = \frac{\sqrt{3}}{2} \qquad \cos 60 = \frac{1}{2}$$

$$\sin 30 = \frac{1}{2} \qquad \sin 60 = \frac{\sqrt{3}}{2}$$

$$\tan 30 = \frac{1}{\sqrt{3}} \qquad \tan 60 = \sqrt{3}$$

Recall that radians are defined by the equivalence

There are 2π radians in a circle.
2π radians is equivalent to 360°.

Example (1) continued

(*c*) Express the solutions to part (*b)* using radians to measure angles instead of degrees.

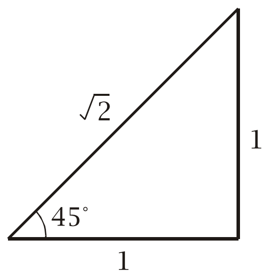
Solution

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

The second special triangle is an isosceles triangle with a base angle of 45° (or $\pi/4$ radians).



Example (2)

From this triangle it is possible to find the exact values of $\sin 45$, $\cos 45$ and $\tan 45$. Express your solutions also in radians.

Solution

$\cos 45 = \sin 45 = 1/\sqrt{2}$ $\qquad$ $\tan 45 = 1$

In radians

$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\qquad$ $\tan\frac{\pi}{4} = 1$

We can combine this knowledge with knowledge of the trigonometric functions and how the values of $\sin x$, $\cos x$ and $\tan x$ vary with the angle x.

Example (3)

Find the exact values of

(a) $\sin 225°$ $\qquad$ (b) $\cos 150°$ $\qquad$ (c) $\tan\left(\frac{5\pi}{6}\right)$ $\qquad$ (d) $\sin\left(-\frac{\pi}{3}\right)$

Solution

(a) 225° lies in the third quadrant, in which $\sin x$ is negative.

$225° = 180° + 45°$

$\sin 225° = -\sin 45° = \frac{-1}{\sqrt{2}}$

(b) $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$

(c) $\tan\left(\frac{5\pi}{6}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

(d) $\sin\left(-\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

The relationship between sides given by Pythagoras's Theorem suggests other "special" triangles, and this is what we now go on to consider.

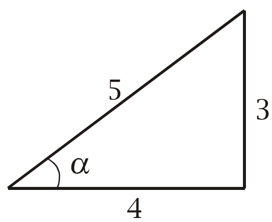
Pythagorean triples and special triangles

The relationship

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

means that a triangle with sides 3, 4, and 5 is a right-angled triangle



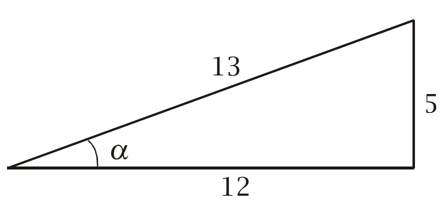
This means that the angle α here is such that

$\tan\alpha = \frac{3}{4}$ $\sin\alpha = \frac{3}{5}$ $\cos\alpha = \frac{4}{5}$

When you see the statement "the angle $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$" in a question you are expected to be able to write down $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$ without recourse to a calculator. The numbers 3, 4, 5 comprise a Pythagorean triple; that is, they are examples of three numbers, *x, y* and *z*, that satisfy the relationship

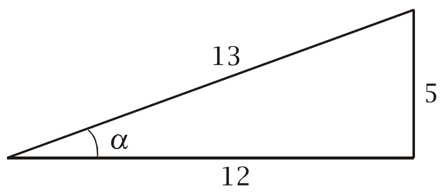
$x^2 + y^2 = z^2$

Another frequently used triangle is the one created by the Pythagorean triple 5, 12, 13. This also creates a right-angled triangle.



Example (4)

Given the triangle



write down exact values of $\tan\alpha$, $\sin\alpha$ and $\cos\alpha$.

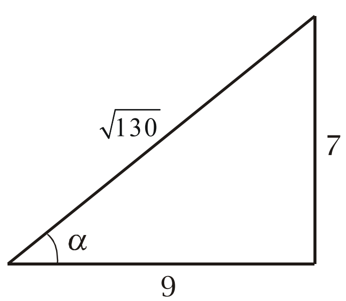
Solution

$\tan\alpha = \frac{5}{12}$ $\sin\alpha = \frac{5}{13}$ $\cos\alpha = \frac{12}{13}$

Such relationships hold in any right-angled triangle, not necessarily a triangle whose sides are in a ratio matching a Pythagorean triple.

Example (5)

Write down the values of $\tan\alpha$, $\sin\alpha$ and $\cos\alpha$ from the following triangle.



Solution

$\tan\alpha = \frac{7}{9}$ $\sin\alpha = \frac{7}{\sqrt{130}}$ $\cos\alpha = \frac{9}{\sqrt{130}}$