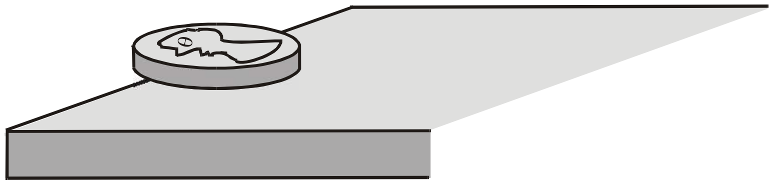
Stability and Oscillations



Ina game of shove- halfpenny a coin is resting on the edge of a table partly overhanging the edge but nonetheless in a stable equilibrium. Its stability is characterised by the fact that it is not falling- that is, it is not losing or gaining gravitational potential energy. In fact, it is the case that for any object (in conservative system) that there is no charge of potential energy.

An object is a subject to conservative forces if its total energy is conserved.

Consider an example, an object such as a planet in orbit, where then energy of the object takes the form of either kinetic energy or potential energy or both. If the system is constant total energy is constant, hence

Kinetic Energy + Potential Energy = Total Energy

$$\frac{1}{2}Mv^2 + U(x) = E$$

Now $\frac{d}{dx}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dx}$

But acceleration is

$$a = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = \frac{dv}{dx}v = v\frac{dv}{dx}$$

So

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx} = a$$

Hence $\frac{d}{dx}\left(\frac{1}{2}mv^2\right) = ma = F$ by Newton's second Law. If an object is in equilibrium the resultant force, F, is zero. Hence

$$\frac{d}{dx}\left\{\frac{1}{2}mr^2 + U(x)\right\} = \frac{d}{dx}\mathrm{E}$$

$$\therefore F + \frac{dU}{dx} = 0$$

$$\therefore \frac{dU}{dx} = 0 \qquad \text{since } F = 0$$

This tells us that an object where there is a possible exchange only between kinetic and potential energy is in a state of equilibrium if, and only if

$$\frac{du}{dx} = 0$$

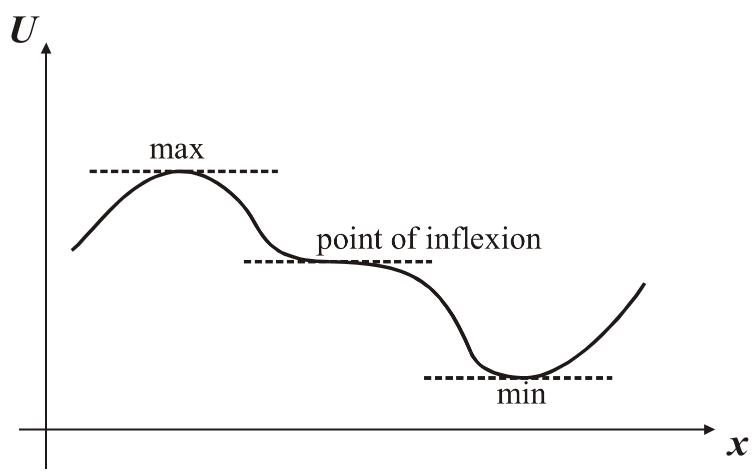
That is, the rate of change of potential energy is zero.

In fact, this result can be generalised to any situation where potential energy is a fraction of a single variable. If a body is subject only to conservative forces, but it is free to move so that its potential energy, U, is a functional of a variable, X, such that $U = U(x)$, then its equilibrium positions are given as solutions to the equation.

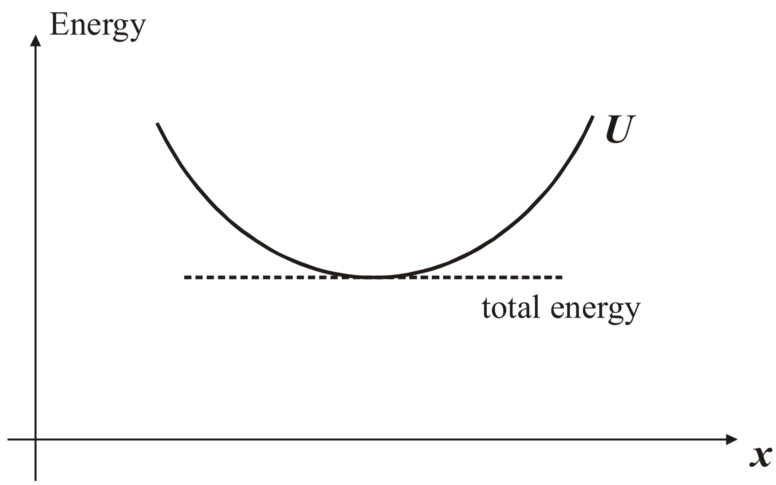
$$\frac{du}{dx} = 0$$

Stability of the Equilibrium

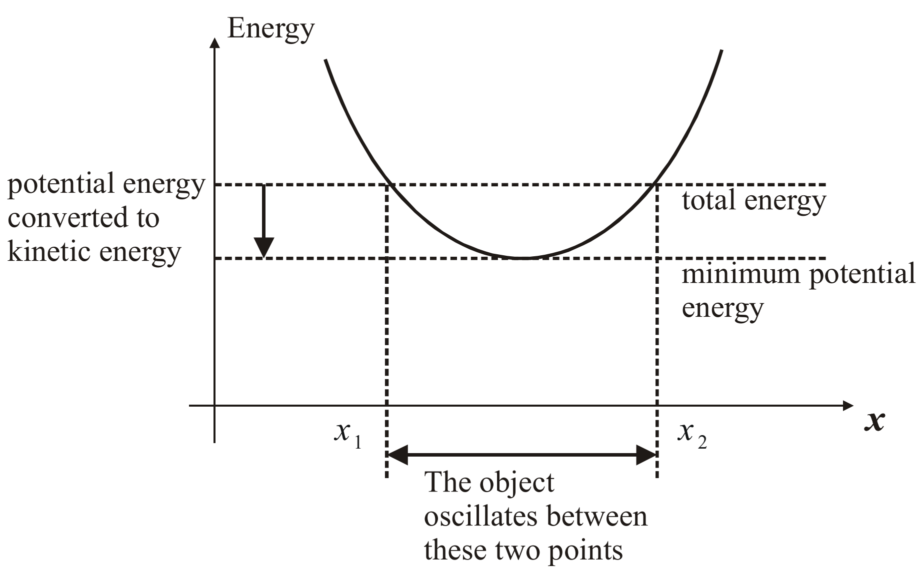
If the potential energy function is a minimum, then the equilibrium point is stable.
If the potential energy function is a maximum then the equilibrium point is unstable.



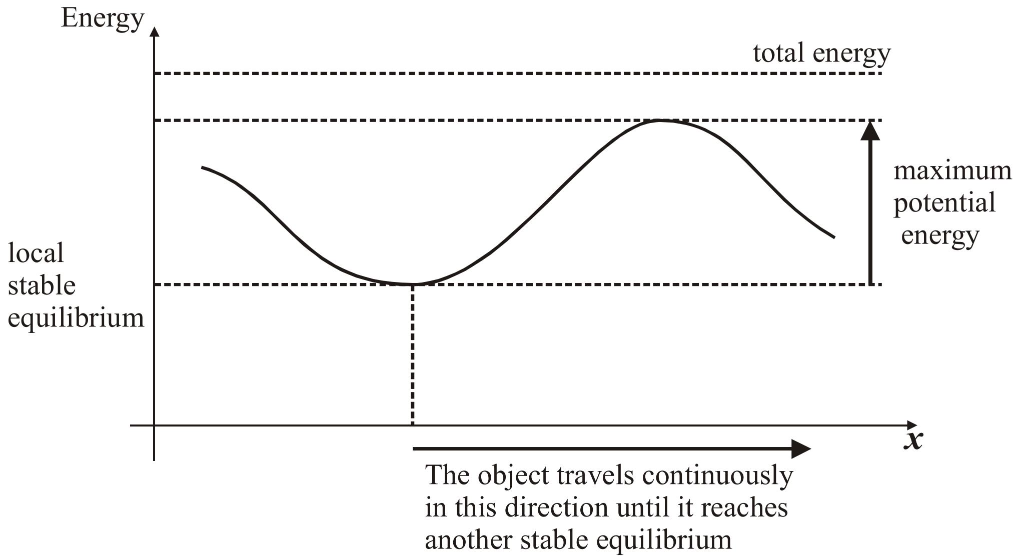
To explain why – when an object moves away from equilibrium its potential energy is converted to kinetic energy. If the object is in static equilibrium at a minimum of the potential energy function then its total energy is equal to its potential energy. Unless it is given energy, say in the form of a push- then it cannot acquire any further energy to get its motion underway.



Even when it is given energy, if the introduction of energy is small (a small push) then it only has energy to “climb” part of the potential energy “hill” and once it has done s it must return to the minimum since the highest point it reaches on the potential energy “slope” is nonetheless, an unstable equilibrium. The object subsequently performs small oscillations about the equilibrium position- that is, assuming hat the system is conservative (after the initial “push” of course) and no further energy can leak in or out.



In order to get the object to leave the stability of the potential energy “well” it must be sufficient energy to overcome a neighbouring maximum.

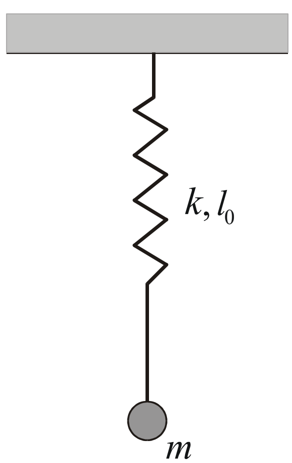


In fact potential energy can also be a function of more than one variable. The case considered so far have been one dimensional cases-that is, cases exhibiting one degree of freedom

Small Oscillations about an equilibrium position

When an object in a stable equilibrium receives a small amount of energy it will oscilate about the stable position. If we assume that the system is conservative, then the object will (approximately or exactly) exhibit simple harmonic motion.

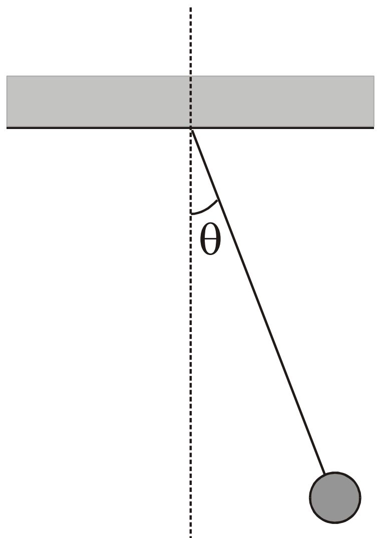
A spring /mass system exhibits this kind of stability.



Its oscillations are given by;

$\ddot{x} = -kx$

where $k = \frac{\lambda}{l_0}$ is the stiffness of the spring (recall that λ is the modulus of the spring if l_0 is its natural length.



A simple pendulum exhibits approximately simple harmonic motion- provided the angle of displacement, σ , is small.

In such systems it is often easier to use the energy method to find the equation of motion- that is, to obtain the energy function and differentiate it.

Use of the second derivative of potential energy.

The equation if motion for some oscillating systems can be obtained from the second derivative of the potential energy function. Assuming that the system is conservative, we have

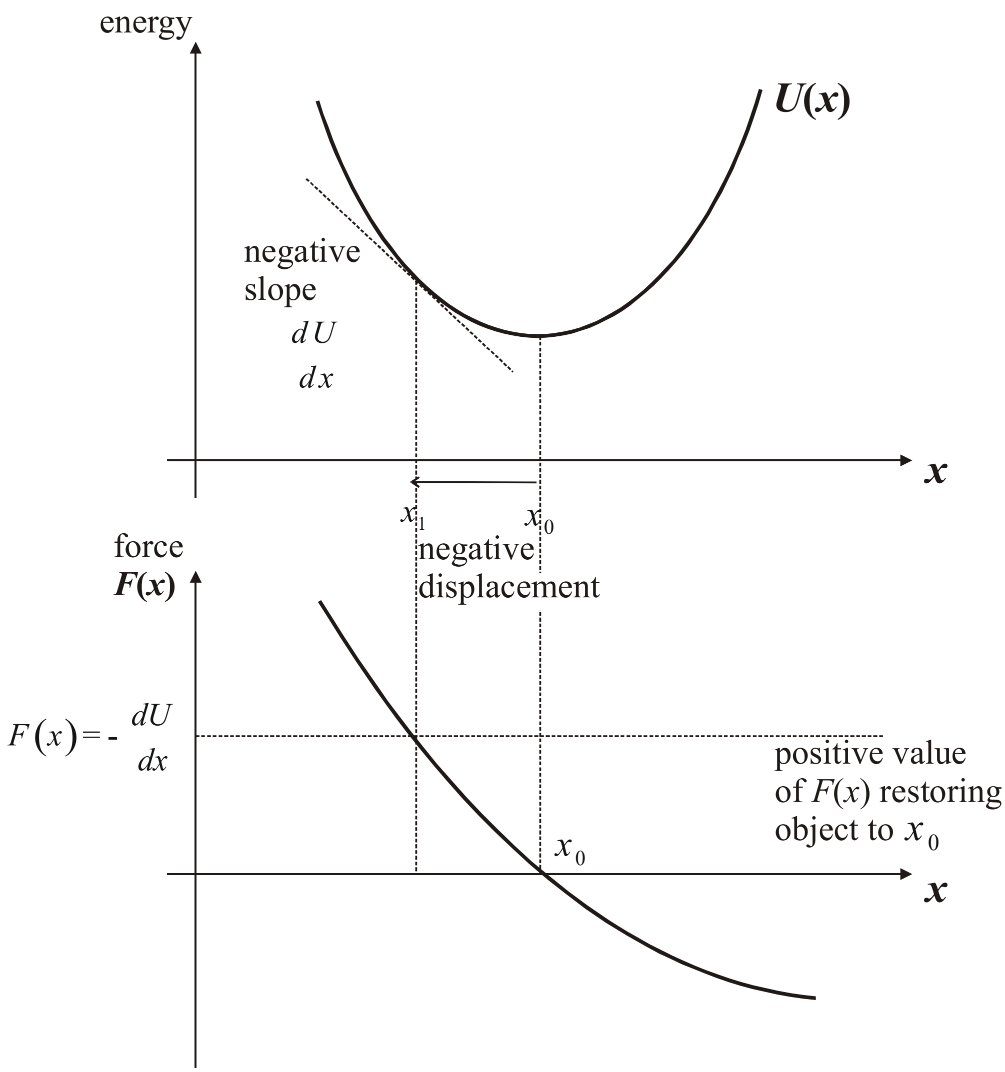
$$m\ddot{x} = \frac{dU}{dx} = F(x)$$

where F is the total force and U is the potential energy.

Let x_0 represent a point of stable equilibrium. If the object is displaced from the stable equilibrium it has been given energy (through an impulse).

However, at the point of stable equations it still experiences zero force. As it moves away from equilibrium it experiences a force that "pushes" it back towards the equilibrium a restorative force. This force acts in the opposite direction to the displacement.

Let x_0 the gradient of the force function $F(x)$ be approximated by a tangent.



The gradient of $F(x)$ is

$F^1(x) \approx$ slope of tangent at x_0

$$\therefore F(x) \approx \frac{F(x)}{x - x_0}$$

Note that the gradient in the diagram is negative. However $(x - x_0)$ is also negative when $x < x_0$ as here, so the equation is correct.

That is $F(x) \approx (x - x_0)F(x)$

Since $F(x) = m\ddot{x}$ we have $m\ddot{x} = (x - x_0)F(x)$

Replacing $F(x) = -U'(x)$ which gives $F'(x) = -U''(x)$ we obtain

$$m\ddot{x} = -(x - x_0)U''(x)$$

as the approximation to the equation of motion of the object about its equilibrium position. This is the equation of simple harmonic motion.

The fact that x_0 represents a stable equilibrium point means that $U''(x_0) \geq 0$
That is, not negative. However it can be equal to zero, in which case the approximation by simple harmonic motion breaks down.

Oscillations involving rotation

If the displacement from stable equilibrium is given in terms of the value of an angle rather that linear displacement you can still use the energy method to find the period of oscillation. However to do so, you need to be able to express kinetic energy in terms of angle.