Static Equilibrium

Scalars and vectors

You should already be familiar with the distinction between *scalars* and *vectors*. A vector is a physical quantity with magnitude (size) and direction. A scalar is a physical quantity with only magnitude.

Example (1)

Which of the following are vectors and which are scalars?

(*a*) Displacement
(*b*) Distance
(*c*) Number
(*d*) Volume
(*e*) Speed
(*f*) Velocity
(*g*) Acceleration
(*h*) Force
(*i*) Weight
(*j*) Gravitational field constant

Solution

(*a*) Displacement is distance travelled in a given direction and is a vector
(*b*) Distance is a scalar
(*c*) Number is a scalar
(*d*) Volume is a scalar
(*e*) Speed is a scalar
(*f*) Velocity is speed in a given direction and is a vector
(*g*) Acceleration is rate of change of velocity and is a vector
(*h*) Force is a vector
(*i*) Weight is the force due to gravity and is a vector
(*j*) Gravitational field constant is a constant for a given gravitational field and is a scalar.

You should also be familiar with the way in which we express vectors in one dimension. A one-dimensional vector acts either forward (+) or backwards (–). We express such differences by using a sign.

Example (2)

A tram is stationed at point *O.* It runs on a horizontal straight track. At first the tram driver takes it 1500 m due East to point *P.* The driver then stops the tram, walks to its other end, and drives it 2750 m due West to point *Q*. Taking *O* as the origin and due East as the positive direction. Express as vectors:

(*a*) The displacement of the tram during the first part of the journey from *O to P.*

(*b*) The displacement of the tram during the second part of the journey from *P* to *Q*.

(*c*) The overall displacement of the tram, which is the vector *O* to *Q*.

Solution

(*a*) We write this as $\overrightarrow{OP}$ and it is given by $\overrightarrow{OP} = 1500$. It is understood that when there is no sign in front of a vector then the sign is positive. That is $\overrightarrow{OP} = 1500 = +1500$.

(*b*) $\overrightarrow{PQ} = -2750$

(*c*) $\overrightarrow{OQ} = 1500 - 2750 = -1250$

Example (2) continued

Write an equation representing the overall journey involving the displacement vectors $\overrightarrow{OP}$, $\overrightarrow{PQ}$ and $\overrightarrow{OQ}$.

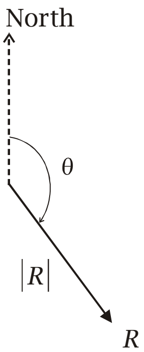
Solution

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\ &= 1500 + (-2750) \\ &= -1250\end{aligned}$$

As this equation indicates it is probably better to see the equation as *adding* the *negative* vector $\overrightarrow{PQ}$ rather than *subtracting* the (positive) *size* of the vector $\overrightarrow{PQ}$ from the vector $\overrightarrow{OP}$. In other words, the minus (–) sign in front of $\overrightarrow{PQ}$ expresses its direction. Thus we use $\overrightarrow{PQ}$ to stand for the displacement vector from *P* to *Q* and in this question this vector is in the opposite sense to the positive direction so it has a *negative* sign.

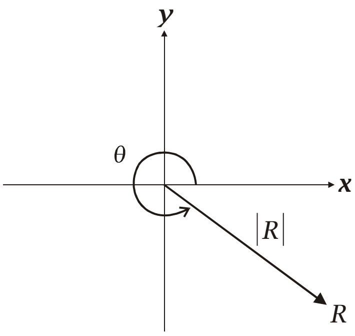
Vectors in two dimensions

In *two dimensions* the use of a sign (+ or –) no longer suffices to determine the direction of a vector. We need two numbers to express the two-dimensional nature of such a vector. In navigation a vector is often specified by means of *range* plus *bearing*. The range is the distance of the tip of the vector from the origin or point of reference, and the bearing is an angle less than 360º measured clockwise from the north direction.



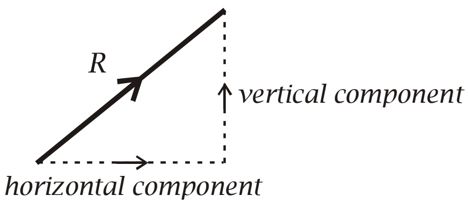
In this diagram the vector is represented by R, its bearing by θ and the size of the vector by $|R|$. The Greek letter, θ, pronounced "theta" is used to denote the bearing. The use of Greek letters to represent quantities is very common in mathematics. However, any letter would do. $|R|$ is read "the modulus of R" or, alternatively, "the size of R". The straight lines are used to denote the modulus of a number, which is the positive size of the number regardless of whether it is a positive or negative number.

In navigation the North direction is the most obvious reference direction, so is used for a bearing, but in coordinate geometry the most common reference direction is the positive x-axis (or equivalent) and angles are usually measured in an *anti-clockwise* direction. The angle is called the *argument* of the vector and the size is called its *modulus*.



Component representation of vectors

Vectors can be resolved into horizontal and vertical *components*.



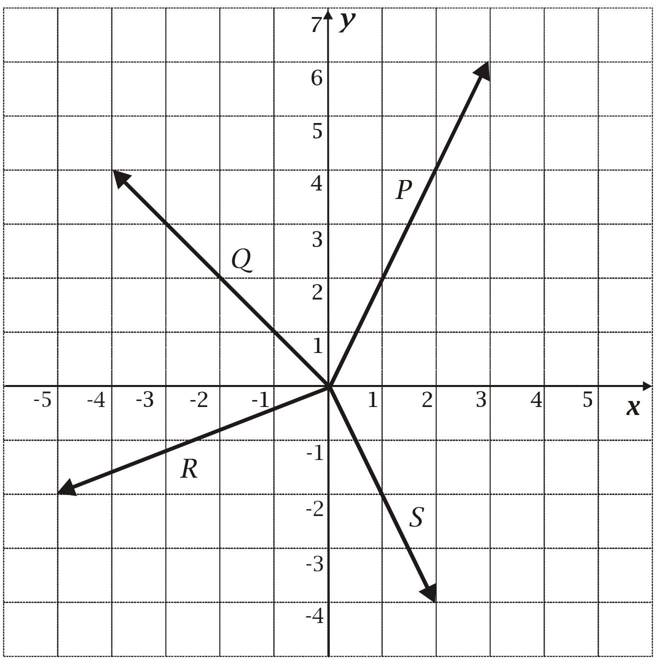
As the above diagram shows, we can think of a vector as the sum of its horizontal and vertical components.

R = horizontal component of R + vertical component of R .

The horizontal and vertical components are themselves one-dimensional vectors, meaning they can take positive or negative values, as the following example illustrates.

Example (3)

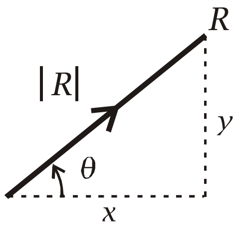
Examine the following diagram showing four vectors *P*, *Q*, *R* and *S*. Write down the horizontal and vertical components of each of these vectors.



Solution

Vector	Horizontal component	Vertical component
P	3	6
Q	–4	4
R	–5	–2
S	2	–4

If x is the horizontal component of a vector and y is its vertical component, then we can use (1) Pythagoras's theorem to find its modulus (size) $|R|$, and (2) trigonometry to find its argument θ.



$$|R| = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

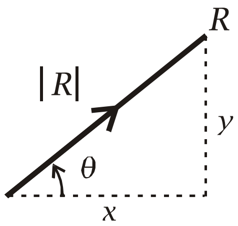
Example (3) continued

Find the modulus and argument of each of the vectors *P*, *Q*, *R* and *S*.

Solution

Vector	Horizontal component	Vertical component	Modulus	Argument
	x	***y***	$\|R\| = \sqrt{x^2 + y^2}$	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
P	3	6	$\|R\| = \sqrt{(3)^2 + (6)^2} = 3\sqrt{5}$	$\theta = \tan^{-1}\left(\frac{6}{3}\right) = 63.4°$
Q	–4	4	$\|R\| = \sqrt{(-4)^2 + (4)^2} = 4\sqrt{2}$	$\theta = \tan^{-1}\left(\frac{4}{-4}\right) = -135°$
R	–5	–2	$\|R\| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$	$\theta = \tan^{-1}\left(\frac{-2}{-5}\right) = 201.8°$
S	2	–4	$\|R\| = \sqrt{(2)^2 + (-4)^2} = 2\sqrt{5}$	$\theta = \tan^{-1}\left(\frac{-4}{2}\right) = 296.6°$

Regarding the arguments in this solution, recall that the argument is an angle between 0° and 360° measured anti-clockwise from the *x*-axis. We can also reverse this process, so that given the modulus and argument of a vector we can find its horizontal and vertical components by means of trigonometric relationships.



$x = R\cos\theta$
$y = R\sin\theta$

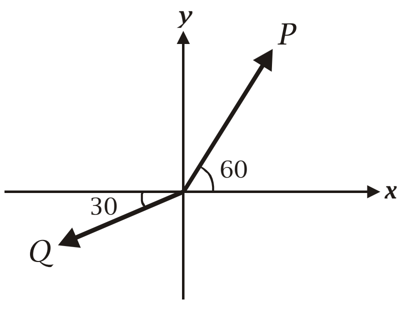
Note that in these last two equations we have not used the symbol $|R|$ for the size of the vector *R*. This is because it is clear from context what is meant. Possibly it would be better if there were always a clear distinction between a vector *R* and its size $|R|$, but it is common in questions dealing with forces to *omit* this distinction, so we also follow that practice here as well. So the symbol *R* is ambiguous. In a given context it may represent a vector, or alternatively the size of that vector, which is a scalar.

Example (4)

Find the horizontal and vertical components of the following vectors.

Vector	Modulus	Argument
P	5	60°
Q	4	210°

Solution



Vector	Modulus	Argument	*x*	*y*
	R	θ	$x = R\cos\theta$	$y = R\sin\theta$
P	5	60°	$x = 5\cos 60 = 2.5$	$y = 5\sin 60 = 4.3$
Q	4	210°	$x = 4\cos 210 = -3.5$	$y = 4\sin 210 = -2$

In this solution the components are given to 2 significant figures.

Addition of Vectors

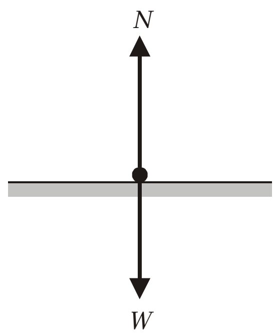
As the last example illustrates adding vectors in component form obeys the obvious rule that you add the components of the vectors separately. In the context of forces this means that we add the horizontal and the vertical components separately. This process is called *resolving horizontally and vertically*.

Resultant forces

When you are adding forces you are finding the *resultant* of the force. This is the single force that is equivalent to all the forces acting at a point. You should already be familiar with this idea in the case of one-dimensional applications.

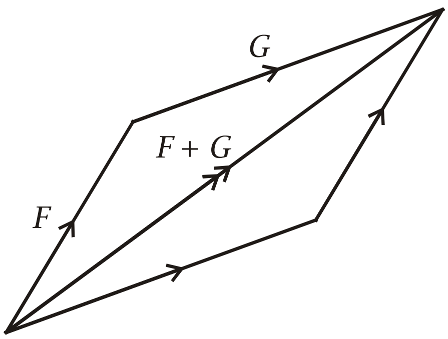
Example (6)
A ball bearing is at rest on a horizontal table. Draw a diagram showing the forces acting on the ball bearing and explain in terms of the resultant of these forces why the ball bearing is not moving relative to the table.

Solution



The ball bearing is subject to two forces. Its weight (*W*) pulling it downwards and a normal reaction from the surface of the table (*N*) pushing it upwards. These two forces cancel each other out – their resultant is zero – and that is why the book, relative to the table, is not moving. $W + N = 0$.

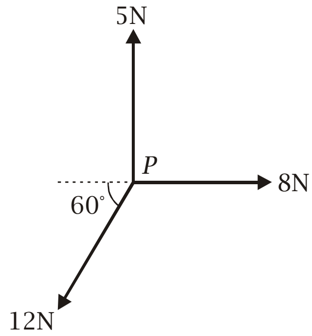
In this context we are applying the idea of a resultant force to the two-dimensional case. The vector $F + G$ is called the resultant of *F* and *G*. The rule for adding vectors that we have just seen means that the resultant obeys the *parallelogram rule*: the resultant is the diagonal of the parallelogram formed with the vectors *F* and *G* as sides.



This is also called the *triangle law of addition* for vectors. By convention a resultant is shown with double arrows. This shows that it is not an *additional* force but the *result* of adding together two (or more) other forces. A vector is specified by its size and direction only, so any two parallel sides of the parallelogram represent the same vector. The diagram shows these forces as sides of a parallelogram, but this should not confuse you into thinking the forces literally form the sides of a parallelogram. The forces all act on a single point. The diagram only shows how the resultant is found. Any collection of forces acting at a single point all lying on the same plane can be added (or "resolved") by adding their components. When forces lie in the same plane they are said to be *coplanar*. In questions the term *coplanar* means that the problem can be solved by resolving in two dimensions and three dimensional aspects can be ignored. The term *horizontal* may also be added. This is to ensure that the plane is lying perpendicular to the force of gravity, so it is a way of saying that gravity can be ignored. So the only forces that need to be resolved are those given in the question.

Example (7)

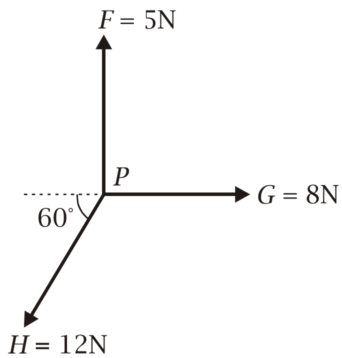
Three horizontal coplanar forces have magnitudes 5 N, 8 N and 12 N and act at the point *P* n the directions shown in the diagram.



Find the magnitude of the resultant and the angle θ measured clockwise made by this resultant with the force of 5N.

Solution

For the sake of clarity let us call these three forces *F*, *G*, and *H* respectively.

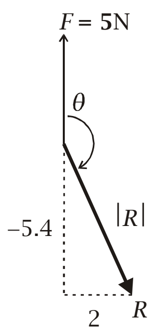


Resolving horizontally

$(\rightarrow)\qquad 8+(-12\cos 60)=8-6=2$

Resolving vertically

$(\uparrow)\qquad 5+(-12\sin 60)=5-10.4=-5.4\ (2\text{ s.f.})$



The magnitude of the resultant is

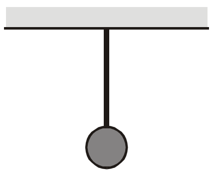
$$|R| = \sqrt{2^2 + (-5.4)^2} = 5.8 \text{ (2 s.f.)}$$

The angle θ is

$$\theta = 180° - \tan^{-1}\left(\frac{5.4}{2}\right) = 110° \text{ (nearest °)}$$

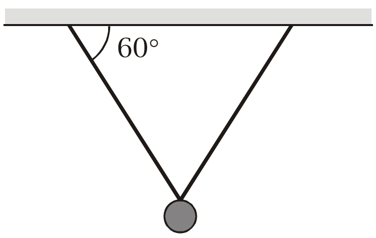
Static equilibrium

Imagine a motionless sphere hanging by a cable suspended from the ceiling.



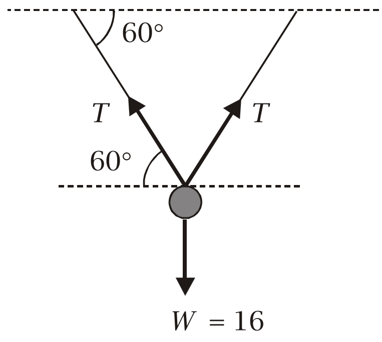
We say that the sphere is in *static equilibrium*. It is *static* because, relative to the ceiling, the sphere is not moving. It is in *equilibrium* because the resultant forces acting on it are zero. The fact that the resultant force is zero enables us to make deductions about the forces acting on the sphere. We can say that the tension in the cable is equal to the weight of the sphere, because the two forces are opposite and cancel one another out. If a static sphere is being pulled to the side by a force *F*, then once again we can make deductions about the forces acting on the sphere.

Example (7)
The diagram shows a sphere of weight 16 N suspended by means of a light cord to the ceiling running through a smooth hook. The cord makes an angle of 60° with the horizontal ceiling. Find the tension in the cord.

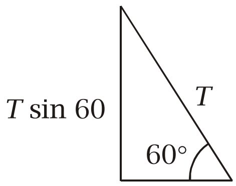


Solution
The question uses the term *light*. This is introduced to indicate that we can ignore the weight of the cord. The term *smooth* indicates that the friction at the hook is zero – it too

can be ignored. We begin by making a new diagram showing the forces acting on the sphere: the weight of the sphere given as 16 N and the tension T in the cord.



As the diagram indicates the magnitude of the tension in the two parts of the cord is equal. Strictly these tensions are vectors, but here we use T to indicate the magnitude of the tension. This magnitude must be equal in both cords because if it were not there would be a resultant horizontal force acting on the sphere and the sphere would not be in static equilibrium. The vertical components of the tension in the *two* parts of the cord combine to equal the weight.



Therefore, resolving vertically

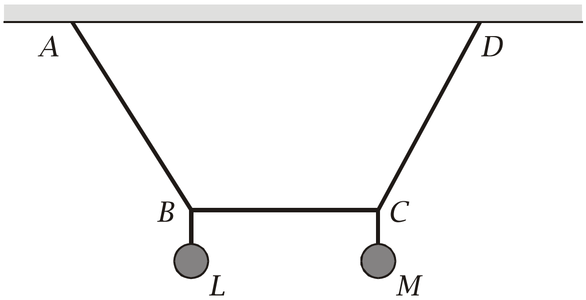
$2T\sin 60 = 16$

$T = \dfrac{16}{2\sin 60} = 9.23... = 9.2\text{ N (2.s.f.)}$

To solve this problem we employed the aid of *two* diagrams. One (given in the question), showing the position of the cords and the other (which we drew) showing the forces. The tension acts in the cord so the angle the tension makes with the horizontal is the same as the angle made by the tension. However, the length of the cord has no influence on the size of the tension. You should not confuse the two diagrams. It is a mistake to think that the length of the cord is a clue as to the size of the tension acting in it.

Example (8)

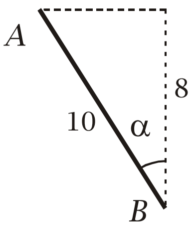
The diagram shows two spheres L and M of weight 12 N and 6 N respectively suspended in static equilibrium by means of light, inextensible cables attached to the spheres at B and C and to the horizontal ceiling at A and D. The length of the cable AB = 10 cm, the cable BC is horizontal and the point B lies 8 cm below the ceiling. The joints at B and C are frictionless.



Find the length of the cable CD and the tension running in this cable.

Solution

The information in the question enables us to find the angle at which the cable AB is hanging.



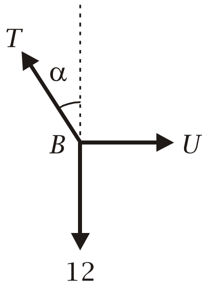
The diagram above indicates that the angle made by AB with the vertical is given by

$$\cos\alpha = \frac{4}{5}$$

By Pythagoras's theorem the side opposite is 6 cm and we also have

$$\sin\alpha = \frac{3}{5}$$

Let T be the tension in the cable AB and U be the tension in the cable BC. The forces acting at B are shown in the following diagram.



Resolving vertically

$(\uparrow)\qquad T\cos\alpha = 12$

$$T \times \frac{4}{5} = 12$$

$$T = 15\text{ N}$$

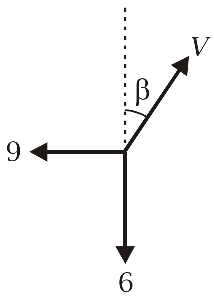
Resolving horizontally

$(\rightarrow)\qquad U = T\sin\alpha$

$$= 15 \times \frac{3}{5}$$

$$= 9\text{ N}$$

Let the tension acting in the cord *CD* be *V* and the angle the cord *CD* makes with the vertical be β. Then at *C* we have the following force diagram.



This employs the principle that the tension in the cable pulling at *B* must be equal and opposite to the tension in the cable pulling at *C*.

Resolving vertically

$(\uparrow)\qquad V\cos\beta = 6$

Resolving horizontally

$(\rightarrow)\qquad V\sin\beta = 9$

Therefore

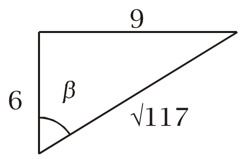
$$\frac{V\sin\beta}{V\cos\beta} = \tan\beta = \frac{9}{6}$$

$$\beta = \tan^{-1}\left(\frac{9}{6}\right) = 56.309... = 56° \ \text{(nearest °)}$$

By Pythagoras's theorem

$$V = \sqrt{9^2 + 6^2} = \sqrt{117} = 10.816... = 11 \text{ N}(2.\text{s.f.})$$

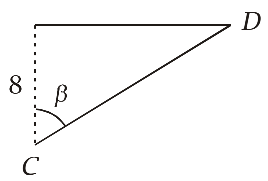
Since angle $\tan\beta = \frac{9}{6}$



$$\cos\beta = \frac{6}{\sqrt{117}}$$

The length CD is such that

$$\cos\beta = \frac{8}{CD}$$



$$CD = \frac{8}{\cos\beta} = \frac{8}{\left(\frac{6}{\sqrt{117}}\right)} = \frac{4}{3}\sqrt{117} = 14.422... = 14 \text{ cm} \ (2.\text{s.f.})$$

Alternatively

$$CD = \frac{8}{\cos\beta} = \frac{8}{\cos(56.309...)} = 14.422... = 14 \text{ cm} \ (2.\text{s.f.})$$