Stationary points, higher derivatives and curve sketching

Second and Higher Derivatives

The *second derivative* of a function is the derivative of the first derivative. The *third derivative* is the derivative of the second derivative, and so forth. Given a function $y = f(x)$, its first, second and third derivatives are written

First derivative	$f'(x) = \frac{dy}{dx}$
Second derivative	$f''(x) = \frac{d^2y}{dx^2}$
Third derivative	$f'''(x) = \frac{d^3y}{dx^3}$

We can combine the symbols in any meaningful way. For example, in

$$f''(x) = \frac{d}{dx}f'(x)$$

$f''(x)$ denotes the second derivative and the expression $\frac{d}{dx}$ indicates the process of taking a derivative; $\frac{d}{dx}f'(x)$ indicates the process of taking the next derivative of $f'(x)$.

Example (1)

Find the first, second and third derivatives of the function $f(x) = x^4 - 2x^3 + \frac{1}{x}$

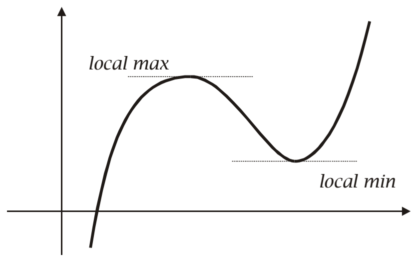
Solution

$$f(x) = y = x^4 - 2x^3 + x^{-1}$$
$$f'(x) = \frac{dy}{dx} = 4x^3 - 6x^2 - x^{-2} = 4x^3 - 6x^2 - \frac{1}{x^2}$$
$$f''(x) = \frac{d}{dx}f'(x) = 12x^2 - 12x + 2x^{-3} = 12x^2 - 12x + \frac{2}{x^3}$$
$$f'''(x) = \frac{d}{dx}f''(x) = 24x - 12 - 6x^{-4} = 24x - 12 - \frac{6}{x^4}$$

For higher derivatives, $f^{(n)}(x)$ denotes the *n*th derivative, and $f^{(n)}(x) = \frac{d}{dx}f^{(n-1)}(x)$.

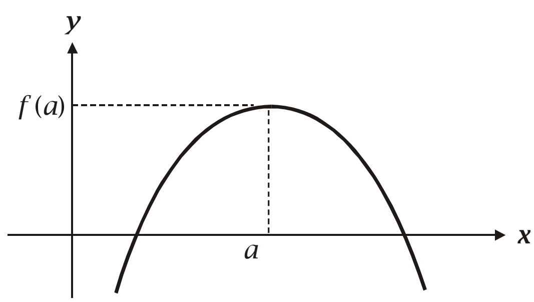
Stationary Points

Mathematicians and physicists are interested in points where a function $y = f(x)$ takes a local maximum or minimum.



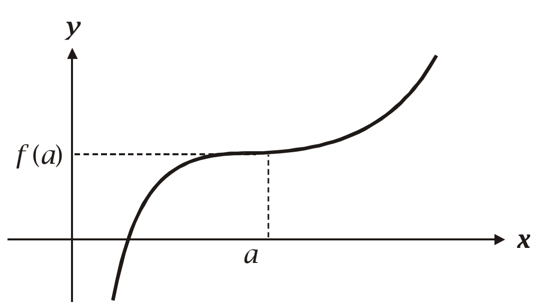
A function with a local maximum and minimum

A local maximum at a means that in the region just around a the greatest value of $y = f(x)$ occurs at $f(a)$.



A local maximum

But a local maximum does not mean that the function cannot take higher values elsewhere, outside the local region. That is why it is called a *local* maximum, because another maximum could be greater or the function could increase elsewhere without limit. A local maximum is an example of a *turning point*. There are three kinds of turning points – local maximum, local minimum or point of inflexion. A point of inflexion occurs where the gradient of the function is zero, but the function carries on increasing or decreasing.



A point of inflexion

Turning points are also called *stationary points*, because at them the gradient becomes zero. Hence, the criterion for turning (stationary) points is

$$\frac{dy}{dx}=0 \quad \text{or} \quad f'(x)=0.$$

So for a function $y=f(x)$ to find where turning points occur we solve the equation

$$\frac{dy}{dx}=0 \quad \text{or} \quad f'(x)=0.$$

Example (2)

The function $y=f(x)$ is defined by

$$f(x)=x^3-3x-1$$

Find the stationary points of $f(x)$.

Solution

For turning points, $f'(x)=0$. That is

$$f'(x)=3x^2-3=0$$
$$3(x^2-1)=0$$
$$3(x+1)(x-1)=0$$
$$x=1 \text{ or } x=-1$$

To complete the question we need to find the corresponding y values.

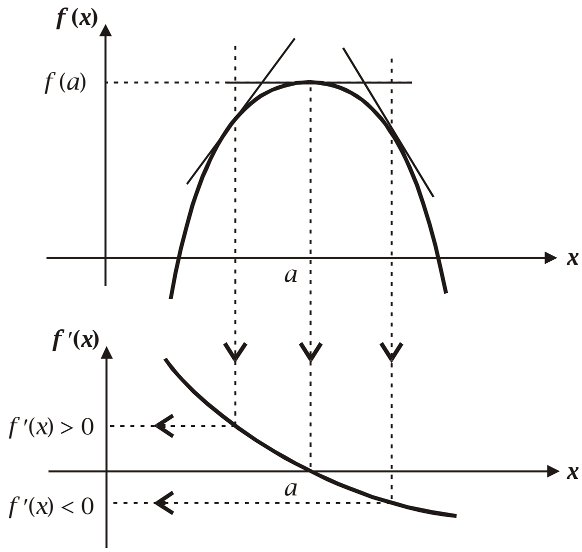
$$y(1)=(1)^3-3\times1-1=-3$$
$$y(-1)=(-1)^3-3\times(-1)-1=1$$

The turning points are at $(1,-3)$ and $(-1,1)$.

This tells where the turning points occur, but for each turning point there remains the question – is it a local maximum, a local minimum or a point of inflexion? We need a further way of determining the character of the turning point.

Determining the character of a stationary point

To answer this question, observe that around a local maximum the gradient is first positive then zero then negative. In the following diagram (next page) the top graph shows a function $y=f(x)$ that has a maximum at the point $x=a$. Before this point is reached $(x<a)$ the slope of the curve (the gradient of its tangent) is directed upwards – meaning it is positive. After the maximum has been passed $(x>a)$ the gradient is directed downwards and is negative.

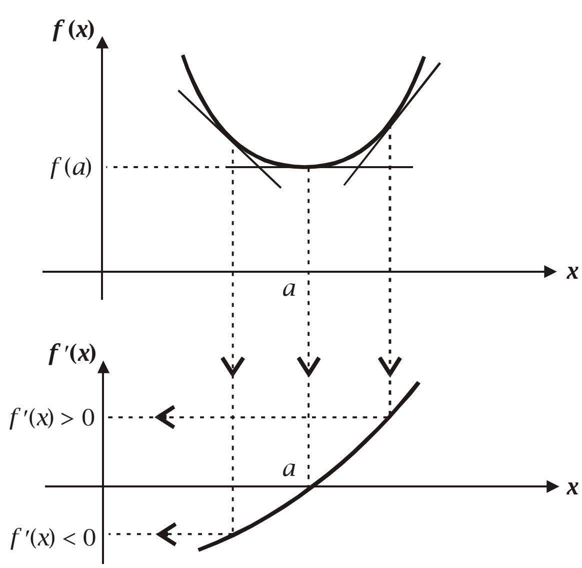


The second graph is a graph of the gradient $\frac{dy}{dx} = f'(x)$. In the region $x < a$, $f'(x)$ is positive; at $x = a$ it is zero, and in the region $x > a$ it is negative. So the graph of $\frac{dy}{dx} = f'(x)$ slopes downwards around a maximum, and does itself have a negative gradient. This provides us with two ways of determining whether a turning point at $x = a$ is a maximum.

Local maximum (1) $f'(x) > 0$ when $x < a$ $\rightarrow f'(a) = 0 \rightarrow f'(x) < 0$ when $x > a$

(2) $f''(a) < 0$

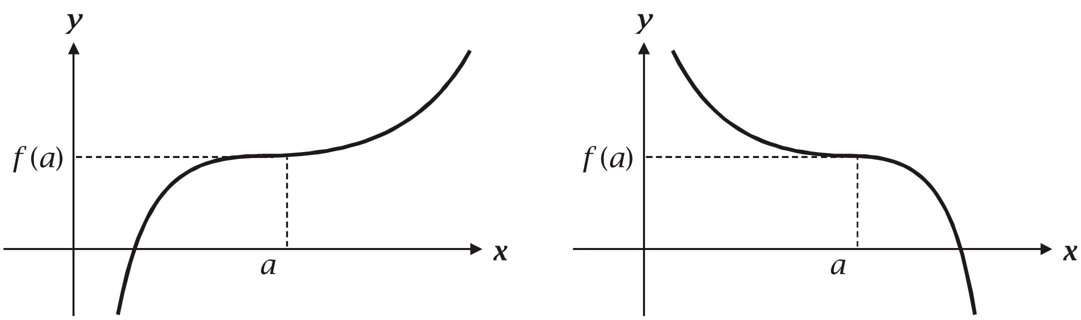
The argument works in reverse for a local minimum.



The diagram illustrates the idea that around a local minimum the gradient of the curve $y = f(x)$ is first negative, then zero, then positive, which also means that the second derivative (the gradient of the gradient) is positive.

Local minimum	(1)	$f'(x) < 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) > 0$ when $x > a$
	(2)	$f''(a) > 0$

A point of inflexion occurs when the gradient becomes zero then returns to being either upward or downward sloping:



So the criterion for a point of inflexion is

Point of inflexion	(1)	$f'(x) < 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) < 0$ when $x > a$ or $f'(x) > 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) > 0$ when $x > a$
	(2)	$f''(a) = 0$

Example (2) continued

The function $y = f(x)$ defined by

$f(x) = x^3 - 3x - 1$

has stationary points at $(1,-3)$ and $(-1,1)$. By finding the second derivative of $y = f(x)$ determine the character of these stationary points, and sketch the graph.

Solution

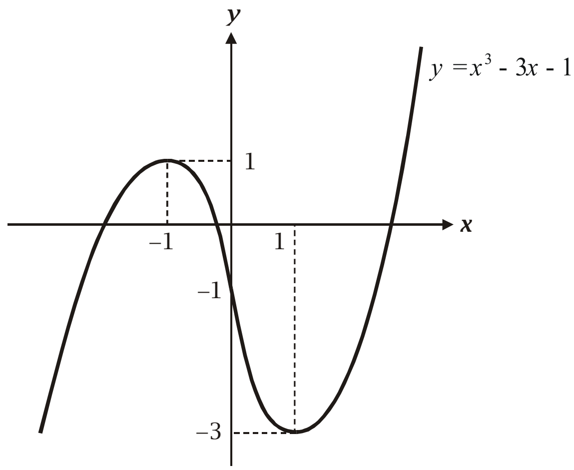
$f'(x) = 3x^2 - 3 = 0$

$f''(x) = 6x$

$f''(1) = 6 > 0 \Rightarrow \text{min}$ $\qquad$ $f''(-1) = 6 < 0 \Rightarrow \text{max}$

So the point at $(1,-3)$ is a local minimum and the point at $(-1,1)$ is a local maximum.

We now have all the information we need to sketch the graph as follows.



Curve sketching

As the previous example illustrates location of stationary points and determination of their character provides all the information required in order to sketch a polynomial function. In summary

1. Locate the stationary points using the criterion that at a stationary point

   $\frac{dy}{dx} = 0$ or $f'(x) = 0$

   Substitute back into $y = f(x)$ to find the y coordinates of the stationary points.

2. Determine the character of each stationary point using the criteria

   Local maximum

   (1) $f'(x) > 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) < 0$ when $x > a$

   (2) $f''(a) < 0$

   Local minimum

   (1) $f'(x) < 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) > 0$ when $x > a$

   (2) $f''(a) > 0$

   Point of inflexion

   (1) $f'(x) < 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) < 0$ when $x > a$
   or
   $f'(x) > 0$ when $x < a \rightarrow f'(a) = 0 \rightarrow f'(x) > 0$ when $x > a$

   (2) $f''(a) = 0$

3. Sketch the graph by marking on the stationary points first and their character. Between stationary points a function must be either always *increasing* or always *decreasing*.

Further notes

1. Whilst turning points and their character are all you need to know in order to sketch basic polynomial functions, other information can be useful if only to check your results. For instance, (*a*) roots of the function provide points where the function crosses the *x*-axis; these can be found by factorising. (*b*) Substituting $x = 0$ gives the point where the function crosses the *y*-axis. (*c*) Looking at large positive and negative *x* values shows the character of the function as *x* approaches $+\infty$ or $-\infty$.
2. For each type of stationary point we provide *two* criteria – one in terms of the second derivative; the other by looking at what happens to the first derivative (gradient function) around the turning point. It turns out that *both* are needed in practice because (*a)* the criterion for the second derivative is usually quicker, but (*b*) sometimes it is very tedious to find the second derivative and using the first criterion is actually easier. Usually, questions at this level will be solved by means of finding the second derivative, but in subsequent chapters the other method is often the better one to use in practice.
3. This chapter deals with *elementary* curve sketching of polynomial functions. The ideas introduced here are of universal validity, but there are other kinds of functions for which the ideas here are not sufficient for sketching their curves – for instance, rational functions that are expressions were one function is divided by another. Techniques for sketching these curves and others are developed in subsequent chapters.

What follows is a harder example.

Example (3)

Sketch the function $f(x) = 2x^5 + 3x^3 - x$

Solution

$f(x) = 2x^5 + 3x^3 - x$

We begin by finding the derivative of this function; we then use this derivative to find its turning points.

$f'(x) = 10x^4 + 9x^2 - 1$

which is a quadratic in x^2, hence

$f'(x) = (10x^2 - 1)(x^2 + 1)$

For turning points, $f'(x) = 0$

$\therefore \quad (10x^2 - 1)(x^2 + 1) = 0$

$10x^2 - 1 = 0$ or $x^2 + 1 = 0$

$(\sqrt{10}x - 1)(\sqrt{10} + 1) = 0$

$x = \dfrac{1}{\sqrt{10}}$ or $x = -\dfrac{1}{\sqrt{10}}$

$x^2 + 1 = 0$ has no real root

At this juncture, we find the coordinates of the turning points.

The y-coordinates of these points are given by substituting these values into $f(x)$

When $x = \frac{1}{\sqrt{10}}$, $y = 2\left(\frac{1}{\sqrt{10}}\right)^5 + 3\left(\frac{1}{\sqrt{10}}\right)^3 - \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left(\frac{1}{50} + \frac{3}{10} - 1\right) = \frac{1}{\sqrt{10}}\left(\frac{1+15-50}{50}\right) = -\frac{17}{25\sqrt{10}}$

When $x = -\frac{1}{\sqrt{10}}$, $y = 2\left(-\frac{1}{\sqrt{10}}\right)^5 + 3\left(-\frac{1}{\sqrt{10}}\right)^3 + \frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}}\left(-\frac{1}{50} - \frac{3}{10} + 1\right) = \frac{1}{\sqrt{10}}\left(\frac{-1-15+50}{50}\right) = \frac{17}{25\sqrt{10}}$

So the coordinates of the turning points are, $\left(\frac{1}{\sqrt{10}}, -\frac{17}{25\sqrt{10}}\right)$, $\left(-\frac{1}{\sqrt{10}}, \frac{17}{25\sqrt{10}}\right)$

Now we need to determine whether the turning points are minima, or maxima.

To find the nature of the turning points we need to look at the second derivative

$f''(x) = 40x^3 + 18x$

When $x = \frac{1}{\sqrt{10}}$, $f''(x) = 40\left(\frac{1}{\sqrt{10}}\right)^3 + 18\left(\frac{1}{\sqrt{10}}\right) > 0$, $\therefore$ min

When $x = -\frac{1}{\sqrt{10}}$, $f''(x) = 40\left(-\frac{1}{\sqrt{10}}\right)^3 + 18\left(-\frac{1}{\sqrt{10}}\right) < 0$, $\therefore$ max

We can now sketch the graph.