# Subgroups

Suppose $(G, \circ)$ is a group – that is, *G* is a set of elements and $\circ$ is a binary operation defined on that set and such that the four group axioms (Identify, Inverses, Closure, Associativity) are satisfied.

Suppose, also, that *H* is a proper subset of *G*, that is $H \subseteq G$. Then *H* will be a subgroup of *G* if the group of axioms also apply to *H* under operation $\circ$. That is if $(H, \circ)$ also satisfies the group of axioms of identify, inverses, closure and associativity.

We prove that $(H, \circ)$ is a subgroup of $(G, \circ)$ by verifying the group properties for *H*. However, if $\circ$ is associative on *G* then $\circ$ must be associative on *H*, because *H* is a subset of *G*. So to show a group is a subgroup we have to check the three properties of (1) Identify, (2) inverses and (3) Closure.

To illustrate the subgroup property, consider the symmetry group $S_3$ of the equilateral triangle. The symmetries are

| | | | |
|---|---|---|---|
| $I$ | identify (zero rotation) | $Q_1$ | reflection in the *y*-axis |
| $R_1$ | rotation by $2\pi/3$ | $Q_2$ | reflection in the line $\pi/6$ |
| $R_2$ | rotation by $4\pi/3$ | $Q_3$ | reflection in the line $-\pi/6$ |

$S_3$ has combination table

| | $I$ | $R_1$ | $R_2$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $R_1$ | $R_2$ | $Q_1$ | $Q_2$ | $Q_3$ |
| $R_1$ | $R_1$ | $R_2$ | $I$ | $Q_2$ | $Q_3$ | $Q_1$ |
| $R_2$ | $R_2$ | $I$ | $R_1$ | $Q_3$ | $Q_1$ | $Q_2$ |
| $Q_1$ | $Q_1$ | $Q_2$ | $Q_3$ | $I$ | $R_2$ | $R_1$ |
| $Q_2$ | $Q_2$ | $Q_3$ | $Q_1$ | $R_1$ | $I$ | $R_2$ |
| $Q_3$ | $Q_3$ | $Q_1$ | $Q_2$ | $R_2$ | $R_1$ | $I$ |

As the partition lines indicate, the set of rotations $\{I, R_1, R_2\}$
Is a subgroup of $S_3$.

To verify that this is a group and hence is a subgroup of $S_3$

(a) Identity

$$\mathrm{I} \in \{\mathrm{I}_1\ \mathrm{R}_1\ \mathrm{R}_2\}$$

(b) Inverses

$$R_1^{-1} = R_2$$
$$R_2^{-1} = R_1$$

(c) Closure

The set is closed

When one group is a subgroup of another we can use the symbol

$$(H, \circ) \le (G, \circ)$$

or more simply

$$H \le G$$

The set of reflections $\{Q_1\ Q_2\ Q_3\}$ is not a subgroup of $S_3$. The set is closed and each element is its own inverse, but the set does not contain the identity.

The null set

The null set, $\varnothing$, is a subset of very set, but it may not be a group under the group binary operation. Therefore, strictly speaking, if we are demonstrating that one set is a subgroup of another group, we should check that the set in question is not the null set.