Summation of finite series using standard results

Summation notation

At this stage you should be familiar with the summation notation. In case you are not, we remind you of the meaning of the expression

$$\sum_{r=1}^{n} u_r$$

This means *sum the series whose general term is* u_r *from* r = 1 *to* r = n. In other words

$$\sum_{r=1}^{n} u_r = u_1 + u_2 + \ldots + u_r + \ldots + u_n$$

Here the term u_r is a *dummy* term that is replaced in context by something specific. For example, in the expression

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + \ldots. + k^2 + \ldots + n^2$$

u_r has been replaced by r^2 and we can now see specifically what the sum means. This gives us a convenient shorthand expression for writing down sums of series, but we wish to find out what those sums are. In this chapter you are *told* what certain sums are and develop skill in using the summation notation by using these *standard results*. The standard results are all proven using the technique of *mathematical induction*, but it is normal to introduce them without justification and first offer the student practice in using them. So the proofs follow in a later chapter.

Standard results for the summation of finite series

We are given the following standard results for finite series:

$$\sum_{r=1}^{n} 1 = n$$

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

By making substitutions into these standard results, we can find specific sums.

Example (1)

Find $\sum_{r=1}^{10} r^2$

In this example we merely substitute for n into the appropriate standard result

$$\sum_{r=1}^{10} r^2 = \frac{1}{6} \times 10(10+1)(2\times 10+1)$$
$$= \frac{1}{6} \times 10 \times 11 \times 21 = 385$$

Example (2)

Find $\sum_{r=4}^{10} r^3$

Here we must see the required sum as the difference of two related sums

$$\sum_{r=4}^{10} r^3 = \sum_{r=1}^{10} r^3 - \sum_{r=1}^{3} r^3$$

The required sum is the sum of all the terms up to 10 minus the sum of the terms up to three. Notice that if we are summing from k to n

$$\sum_{r=k}^{n} u_r$$

Then the sum that we subtract is the sum from 1 to k – 1

$$\sum_{r=k}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{k-1} u_r$$

Now

$$\sum_{r=4}^{10} r^3 = \sum_{r=1}^{10} r^3 - \sum_{r=1}^{3} r^3$$
$$= \frac{1}{4} \times 10^2 \times 11^2 - \frac{1}{4} \times 3^2 \times 4^2$$
$$= 2989$$

Example (3)

Find $\sum_{r=1}^{10} (r(r+3)+1)$

Here we must use algebraic operations to transform the expression u_r into expressions that involve only our standard results.

$$\sum_{r=1}^{10}\left(r(r+3)+1\right)=\sum_{r=1}^{10}\left(r^2+3r+1\right)$$
$$=\sum_{r=1}^{10}r^2+3\sum_{r=1}^{10}r+\sum_{r=1}^{10}1$$
$$=\frac{1}{6}\times10\times11\times21+3\left(\frac{1}{2}\times10\times11\right)+10$$
$$=560$$

Example (4)

Prove that $\sum_{r=1}^{n}r(3+r)=\frac{1}{3}n(n+1)(n+5)$

Here we are asked to find a general expression for the sum of a series of n terms in n. We are not required to arrive at a number.

$$\sum_{r=1}^{n}r(3+r)=\sum_{r=1}^{n}\left(3r+r^2\right)$$
$$=3\sum_{r=1}^{n}r+\sum_{r=1}^{n}r^2$$
$$=3\times\frac{1}{2}n(n+1)+\frac{1}{6}n(n+1)(2n+1)$$
$$=\frac{1}{6}n(n+1)(9+2n+1)$$
$$=\frac{1}{6}n(n+1)(2n+10)$$
$$=\frac{1}{3}n(n+1)(n+5)$$