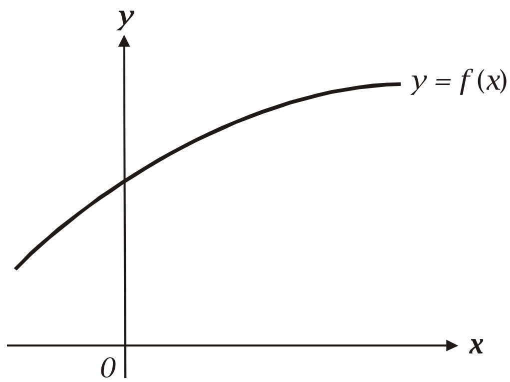
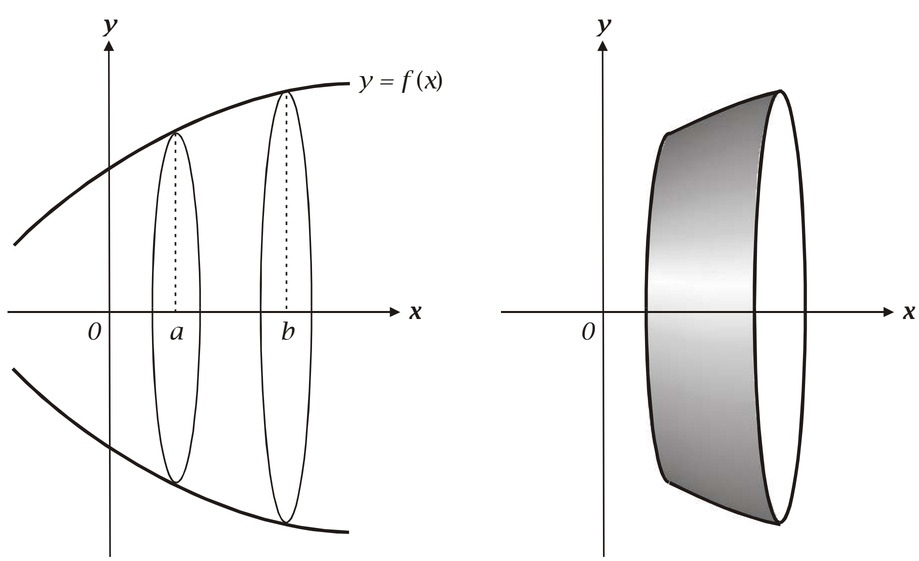
Surface of Revolution

The formula for a surface of revolution

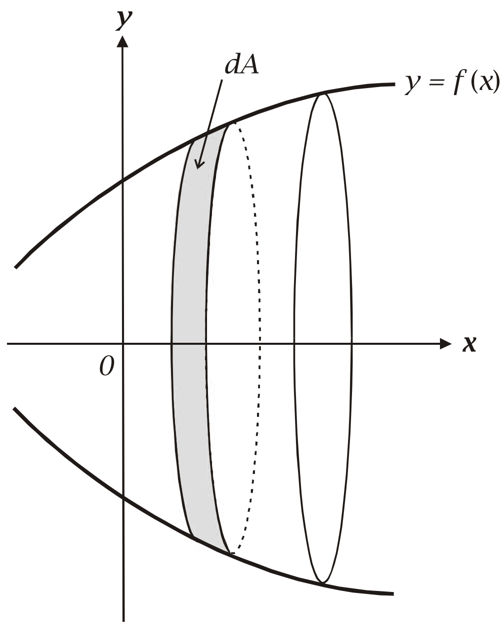
Let us suppose we have a curve given by $y = f(x)$.



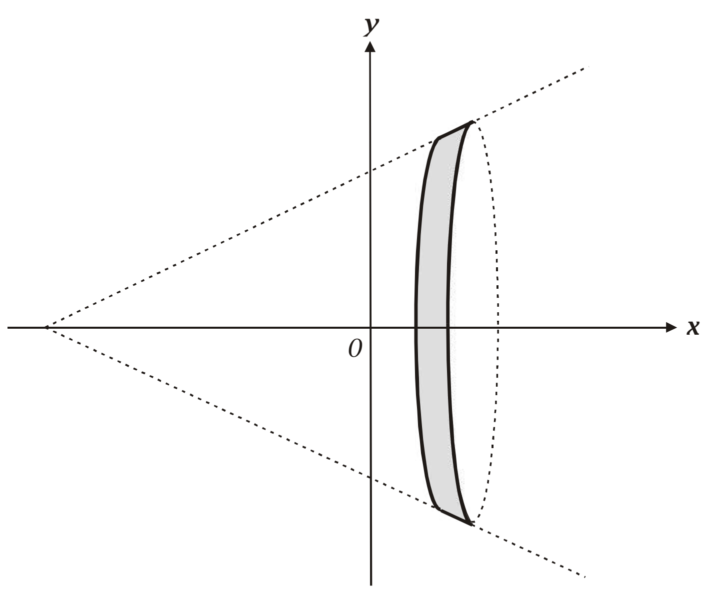
When we rotate this curve around the x–axis, between specific points a and b, we obtain a surface of revolution.



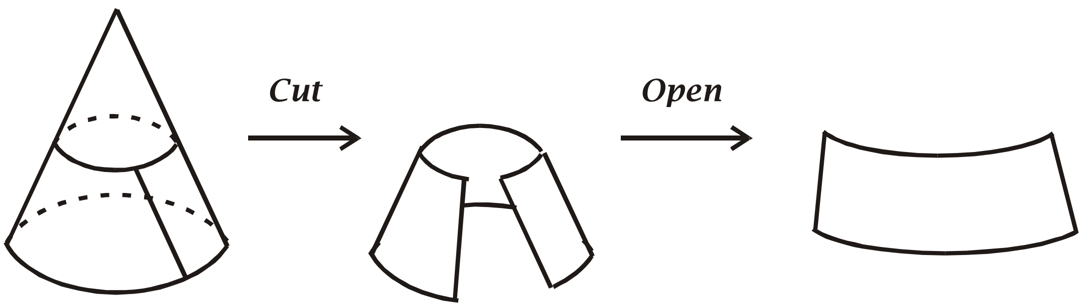
We seek a formula for the area of this surface of revolution. Our approach is the usual one of dividing the surface up into segments and approximating each segment. Firstly, let us divide the surface into cylindrical like portions of area dA .



We now observe that each segment is approximately a segment of a cone.



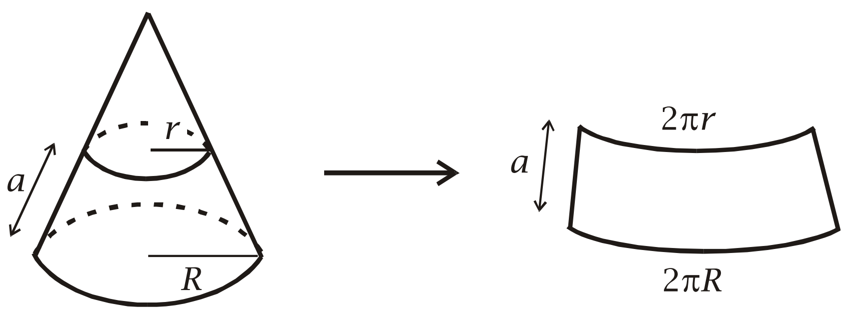
If we now imagine taking this segment of the cone, making a vertical cut in one side of it and opening it out, we would obtain approximately a trapezium.



So the area of the curved surface segment on our original surface of revolution is approximately equal to the area of a curved surface of a right circular cone. The area of a curved surface of a right circular cone is given by

$$\begin{aligned} A &= \tfrac{1}{2}(2\pi r + 2\pi R)a \\ &= \pi a(r+R) \end{aligned}$$

Here r is the radius of the smaller circle and R is the radius of the larger circle.



Now the radius of the curve segment dA is given by

$r = y = f(x)$ and $R = y + dy = f(x + dx)$

and the height of the trapezium is $a = ds =$ length of line segment . Hence each area segment of our surface of revolution is

$$\begin{aligned} dA &\approx \pi ds\big(y + (y + dy)\big) \\ &= \pi ds(2y + dy) \end{aligned}$$

In the limit, as $dx \to 0$, we have $dy \to 0$. Hence

$$dA = 2\pi y\, ds$$

Therefore, the whole surface of revolution is given by the integral of the surface elements.

$$A = \int_a^b dA = \int_a^b 2\pi y\ ds$$

But ds, the small change in the line segment, is given by

$$ds = \left\{ \sqrt{1 - \left(\frac{dy}{dx} \right)^2} \right\} dx \text{ or } ds = \left\{ \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \right\} dy$$

or in parametric form by

$$s = \int \left\{ \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \right\} dt \ .$$

Hence, the surface of revolution is

$$A = 2\pi \int y \left(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right) dx \quad \text{or} \quad A = 2\pi \int y \left(\sqrt{1 + \left(\frac{dx}{dy} \right)^2} \right) dy \quad \text{or} \quad A = 2\pi \int y \left\{ \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} \right\} dt$$

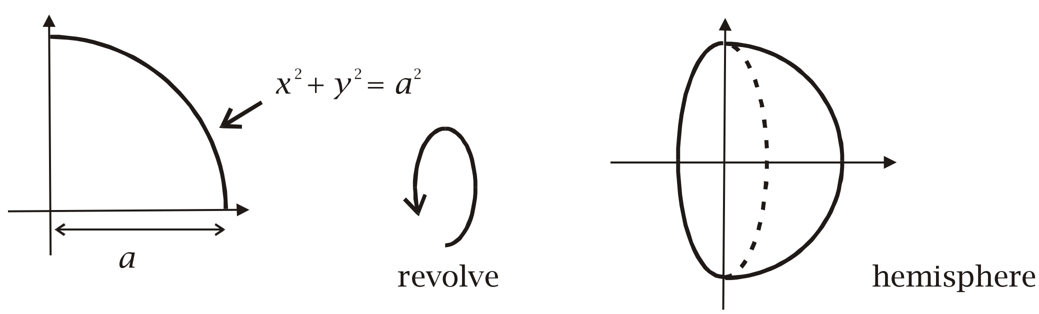
The surface area of a sphere

Example (1)

Find the area of the surface of a sphere of radius *a*.

Solution

The hemisphere is a surface of revolution, obtained by revolving $x^2 + y^2 = a^2$ about the x-axis between 0 and a.



The surface area of a sphere is twice the area of the hemisphere.

Surface of hemisphere = $\int_0^a 2\pi y \ ds$.

We have $x^2 + y^2 = a^2$, hence

$$y = \sqrt{a^2 - x^2}$$
$$\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}$$

Therefore

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$
$$= \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2}\ dx$$
$$= \sqrt{1 + \frac{x^2}{a^2 - x^2}}\ dx$$
$$= \sqrt{\frac{a^2}{a^2 - x^2}}\ dx$$
$$= \frac{a}{y}\ dx$$

Consequently, the surface are of the hemisphere is

$$A = \int_o^a 2\pi y \times \frac{a}{y}\ dx$$
$$= \int_0^a 2\pi a\ dx$$
$$= 2\pi a\left[x\right]_0^a$$
$$= 2\pi a^2$$

The surface area of the sphere is twice this and is given by

Area of surface of a sphere $= 4\pi a^2$

Examples

Example (2)

Find the curved area of the frustum formed by rotating the segment of the line $y = 3x + 4$, between $x = 1$ and $x = 4$ about the *x*-axis.

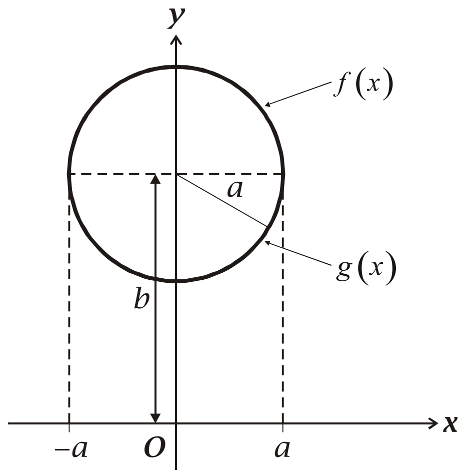
Solution

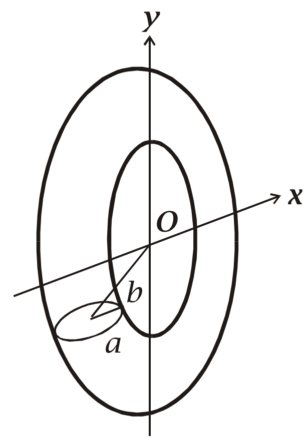
$$y = 3x + 4 \qquad \Rightarrow \qquad \frac{dy}{dx} = 3$$

$$A = 2\pi\int_1^4 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$
$$= 2\pi\int_1^4 (3x + 4)\sqrt{1 + 9}\ dx$$
$$= 2\pi\sqrt{10}\int_1^4 (3x + 4)dx$$
$$= 2\pi\sqrt{10}\left[\frac{3x^2}{2} + 4x\right]_1^4$$
$$= 2\pi\sqrt{10}\left[24 - \frac{3}{2} + 16 - 4\right] = 2\pi\sqrt{10}\,\frac{69}{2} = 69\sqrt{10}\ \pi$$

Example (3)

Find the surface of a torus formed by rotating of the circle $x^2+(y-b)^2=a^2$ about the *x*-axis $b>a$.





$$f(x)=b+\sqrt{a^2-x^2},\quad x\in[-a,a]$$
$$g(x)=b-\sqrt{a^2-x^2},\quad x\in[-a,a]$$
$$\text{area}(\text{torus})=\text{area}(S_\text{f})+\text{area}(S_\text{g})$$

$$\begin{aligned}\text{area}(S_f)&=2\pi\int_{-a}^{a}f(x)\sqrt{1+[f'(x)]^2}dx\\&=2\pi\int_{-a}^{a}\left[b+\sqrt{a^2-x^2}\right]\sqrt{1+\frac{x^2}{a^2-x^2}}dx\\&=2\pi a\int_{-a}^{a}\left(b+\sqrt{a^2-x^2}\right)\frac{1}{\sqrt{a^2-x^2}}dx\\&=2\pi a\int_{-a}^{a}\left(1+\frac{b}{\sqrt{a^2-x^2}}\right)dx\\&=2\pi a\left[2a+b\sin^{-1}\left(\frac{x}{a}\right)\right]_{-a}^{a}\\&=2\pi a\left[2a+b\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right]\\&=2\pi a[2a+b\pi]\end{aligned}$$

$$\begin{aligned}\text{area}(S_g) &= 2\pi\int_{-a}^{a}\left[b-\sqrt{a^2-x^2}\right]\sqrt{1+\frac{x^2}{a^2-x^2}}dx \\ &= 2\pi\int_{-a}^{a}\left(b-\sqrt{a^2-x^2}\right)\frac{a}{\sqrt{a^2-x^2}}dx \\ &= 2\pi a\int_{-a}^{a}\left(\frac{b}{\sqrt{a^2-x^2}}-1\right)dx \\ &= 2\pi a\left[b\pi-2a\right]\end{aligned}$$

Hence

$$\begin{aligned}\text{area}(\text{torus}) &= \text{area}(S_\text{f})+\text{area}(S_\text{g}) \\ &= 2\pi a\left[2a+b\pi+b\pi-2a\right] \\ &= 4\pi^2 ab\end{aligned}$$