# Systems of Linear Equations

## Geometric interpretation of systems of linear equations

Every system of simultaneous equations can be interpreted geometrically. For example, corresponding to the equations:

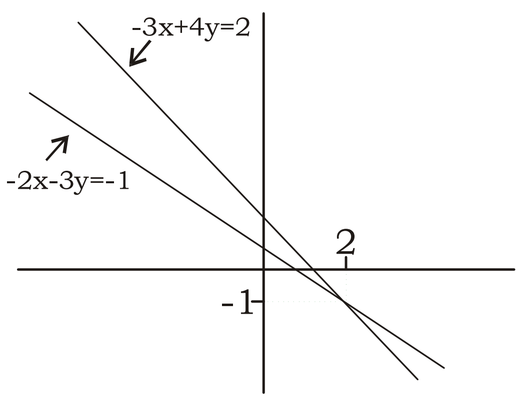
$3x + 4y = 2$

$-2x - 3y = -1$

We can view the solution, $x = 2$, $y = -1$, as the point of intersection of the two lines

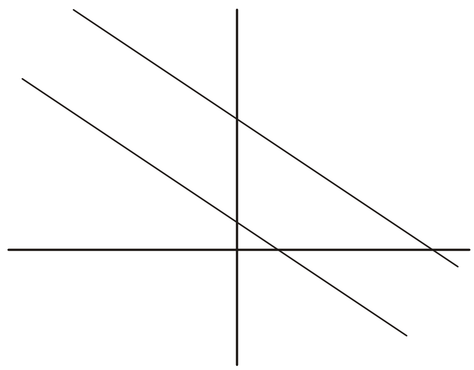
$3x + 4y = 2$

$-2x - 3y = -1$




The system of simultaneous equations will have a unique solution if the graphical representation shows the two lines crossing over to give a point of intersection. However, the graph makes it clear that this need not always be the case.

When the two lines corresponding to two equations are parallel there is no solution to the set of simultaneous linear equations:-



Parallel lines do not give a point of intersection.

In two dimensions two lines are parallel when one is a multiple of another. Thus

$3x + 4y = 2$ (1)

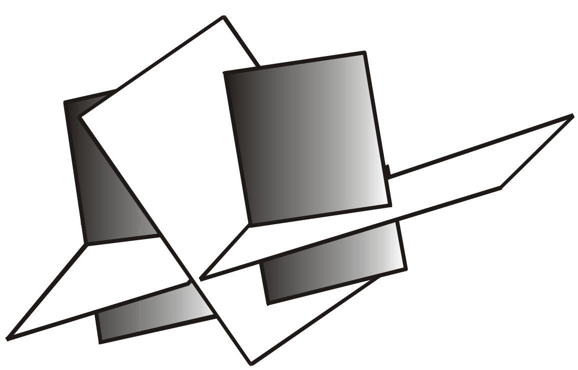
$6x + 8y = 4$ (2)

are parallel since $(2) = (1) \times 2$

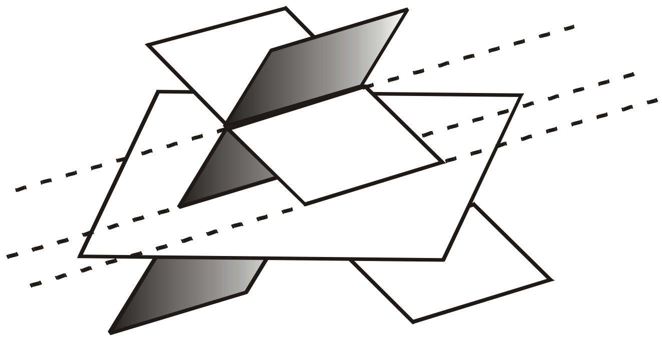
When one equation is a combination of multiples of other equations that equation is said to be linearly dependent on the others. The entire set is linearly dependent. Thus, only linearly independent systems of simultaneous equations have unique solutions. This also applies in three and in *n* dimensions.

We should also explore the visual interpretations of what happens in three dimensions when a system of three equations has and has not a unique solution.

Each equation in the system defines a plane in 3-dimensional space. Thus, for there to be a unique solution, all three planes must intersect at a unique point



The diagram makes it clear that having a unique point of intersection is the exception rather than the rule. One plane may be parallel to another. Alternatively, each plane may intersect each other to give a line, but the lines may not intersect uniquely



## Testing for linear independence

When a set of equations are linearly dependent then one of the equations is a linear sum of multiples of the others.
For example

Determine whether the set of equations

$2x + 2y - 3z = -10$ (1)

$x - y + 2z = 8$ (2)

$8x + 4y - 5z = -14$ (3)

Is linearly dependent.

Solution

If this set is linearly dependent then the third equation would be a linear sum of multiples of the others. Then

$$\alpha\begin{pmatrix}2\\2\\-3\end{pmatrix}+\beta\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}8\\4\\-5\end{pmatrix}$$

For linear dependence there will be non-zero values of $\alpha, \beta$ that make this equation true.

Uncoupling, we have

$2\alpha + \beta = 8$ (1)

$2\alpha - \beta = 4$ (2)

$-3\alpha + 2\beta = -5$ (3)

(1)-(2) gives

$2\beta = 4$

$\beta = 2$

Then in (1)

$2\alpha + 2 = 8$

$2\alpha = 6$

$\alpha = 3$

Check in (3): LHS = -3x3 + 2x2 = -5 = RHS

Which is consistent. Hence this system of equations is linearly dependent since

$(3) = 3\times(1) + 2\times(2)$

When equations are linearly dependent, Gaussian row reduction produces a row with zero elements corresponding to the system

$2x + 2y - 3z = -10$

$x - y + 2z = 8$

$8x + 4y - 5z = -14$

We have the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 2 & -2 & 4 & 16 \\ 8 & 4 & -5 & -14 \end{array}\right)$$

Row reduction results in, for example,:

$$(2)\times 2 \quad \left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 2 & -2 & 4 & 16 \\ 8 & 4 & -5 & -14 \end{array}\right)$$

$$(1)-(2) \quad \left(\begin{array}{ccc|c} 2 & 2 & -3 & -10 \\ 0 & 4 & -7 & -26 \\ 8 & 4 & -5 & -14 \end{array}\right)$$

$$(1)\times 4 \quad \left(\begin{array}{ccc|c} 8 & 8 & -12 & -40 \\ 0 & 4 & -7 & -26 \\ 8 & 4 & -5 & -14 \end{array}\right)$$

$$(1)-(3) \quad \left(\begin{array}{ccc|c} 8 & 8 & -12 & -40 \\ 0 & 4 & -7 & -26 \\ 0 & 4 & -7 & -26 \end{array}\right)$$

$$(2)-(3) \quad \left(\begin{array}{ccc|c} 8 & 8 & -12 & -40 \\ 0 & 4 & -7 & -26 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We can tell that a set of equation is linearly dependent when Gaussian row reduction results in a row with zero elements. Finally, a linearly dependent system will correspond to a matrix with zero determinant.

For the system

$$2x+2y-3z=-10$$

$$x-y+2z=8$$

$$8x+4y-5z=-14$$

The required corresponding square matrix is

$$\begin{pmatrix} 2 & 2 & -3 \\ 1 & -1 & 2 \\ 8 & 4 & -5 \end{pmatrix}$$

With determinant

$$\begin{vmatrix} 2 & 2 & -3 \\ 1 & -1 & 2 \\ 8 & 4 & -5 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ 4 & -5 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ -5 & 8 \end{vmatrix} - 3\begin{vmatrix} 1 & -1 \\ 8 & 4 \end{vmatrix}$$

$$= 2(5-8) + 2(16+5) - 3(4+8)$$

$$= -6 + 42 - 36$$

$$= 0$$

which also shows that the system is linearly dependent.