Techniques of Integration

This is a summary unit. The main techniques of integration are summarised here.

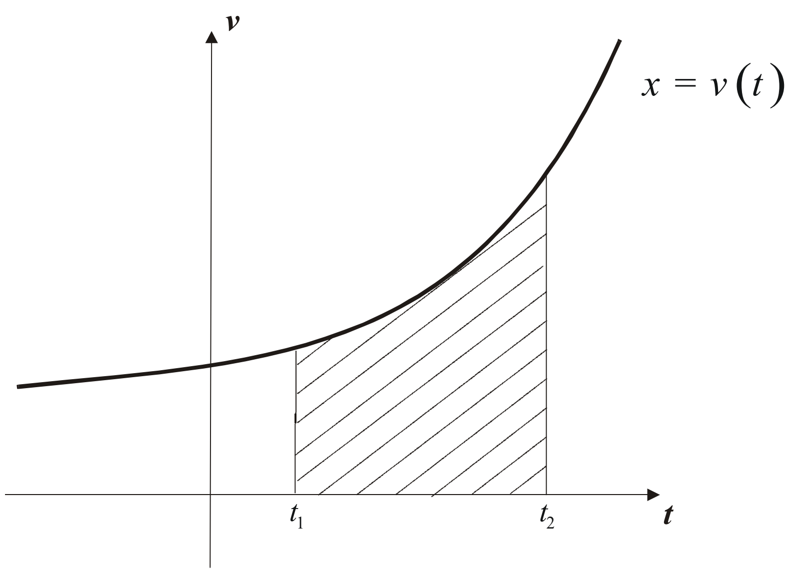
The idea of integration

The idea behind integration derives from the need to find the area under a curve.

Suppose we want to find the distance travelled by an object from time t_1 to t_2. In this case we would be required to find the area under a curve

$$x = v(t)$$

between the points t_1 and t_2



This is called "finding a definite integral" and is represented symbolically by

$$I = \int_{t_1}^{t_2} v(t)\,dt$$

This can be read "the integral of the function. A *definite* integral has limits written on it – that is numbers (or algebraic symbols) specifying the starting and finishing point of the integral. In the above example, these are t_1 and t_2.

An *indefinite* integral takes the form

$$I = \int v(t)\,dt$$

In this form there are no limits.

Integration as the Reverse of Differentiation – Direct Integration

It can be proven that the process of integration is the reverse of the process of differentiation.

$$y = f(x) \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} \frac{dy}{dx} = f'(x)$$

$$G(x) = \int g(x)\,dx \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} g(x)$$

If $f'(x)$ is the derivative of $f(x)$ then $f(x)$ is the integral of $f'(x)$.

We can also express this by

$$G'(x) = g(x) \quad \text{if} \quad \int g(x)\,dx = G(x)$$

This result is called "The fundamental theorem of calculus".

Example

$$\int 2x^3.dx = \tfrac{1}{2}x^4 + c$$

where c is the “constant of integration”.

What this means is that there is a family of functions all of which have the same derivative. The indefinite integral of a function is a family of functions.

Example

$$\int_1^4 3x^2 dx = \left[x^3\right]_1^4 = \left(4^3\right) - \left(1^3\right) = 64 - 1 = 63$$

Integration to infinity

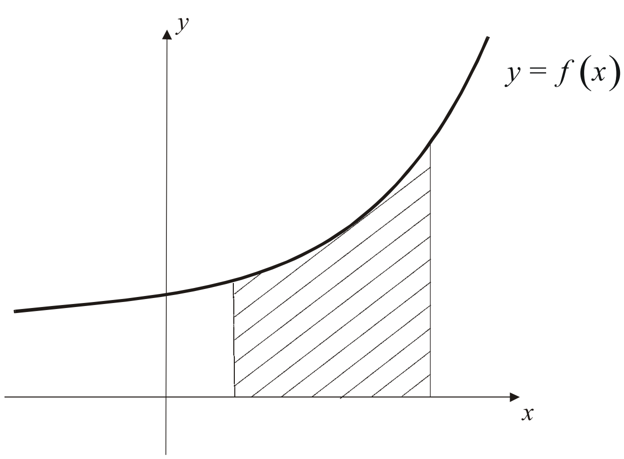
It is possible to evaluate an integral where one or both of the limits is ∞, provided that the integrand *f(x)* tends to some finite limit as *x* tends to either +∞ or –∞ (or both, whatever is appropriate).

Example

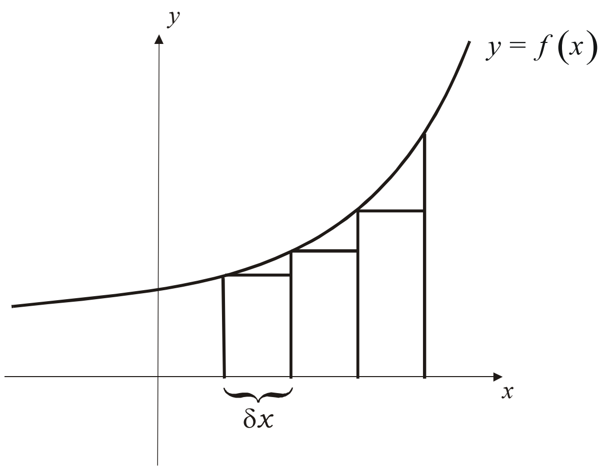
$$I = \int_1^\infty \frac{1}{x} dx = \int_1^\infty x^{-1} dx = \left[-x^{-2}\right]_1^\infty = \left[\frac{-1}{x^2}\right]_1^\infty = 0 - 1 = -1$$

Integration as the Sum of Approximations

We are required to find the area under a given curve, represented by the function $y = f(x)$



We approximate the area by rectangles. Each rectangle will have the same width:



As the rectangles get smaller and smaller – that is, as the width δ*x*, of the rectangle gets smaller – the sum of the area of the rectangles gets closer and closer to the area under the graph.

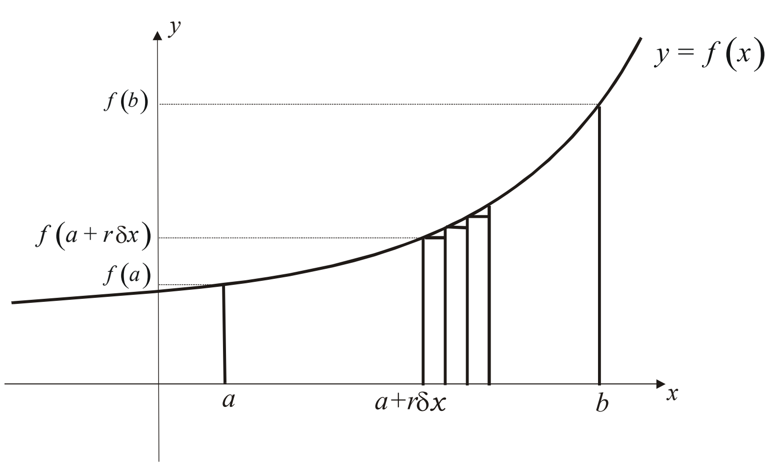
The area of the $r+1th$ rectangle is

$\delta x \times f\left(a + r\delta x\right)$

So the total area is:

Area $= \sum_{r=0}^{n-1} \delta x \times f\left(a + r\delta x\right)$ where there are *n* rectangles

In the limit, as $\delta x \to 0,$ this area becomes equal to the area under the curve.



We denote this limit by:

$$\text{Area } = \int_{a}^{b} f(x)\,dx = \lim_{\delta x \to 0} \sum_{r=0}^{n-1} \delta x \times f(a + r\delta x)$$

The symbol

$$\int_{a}^{b} f(x)\,dx$$

is read "the integral of the function *f(x)* from *a* to *b*.

Direct Integration

Integration is the inverse process of differentiation.

$$\underset{F(x)}{\text{Primitive}} \;\underset{\xleftarrow{\textit{Integrate}}}{\xrightarrow{\textit{Differentiate}}}\; \underset{f(x)}{\text{Derivative}}$$

Some standard integrals found by direct integration are:

function	Integral
$f(x)$	$F(x) = \int f(x).dx$
x^n	$\frac{x^{n+1}}{n+1} + c$
$1/x$	$\ln x + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$

Other functions should also be integrated directly.

Example

$$\int \sec^2 2x = \frac{1}{2}\tan 2x + c$$

Example

If $\frac{dy}{dx} = \sec^2 2x + \text{cosec}^2 3x$ find y

Solution

$$\frac{dy}{dx} = \sec^2 2x + \text{cosec}^2 3x$$
$$\int \left(\sec^2 2x + \text{cosec}^2 3x\right) dx = \int \sec^2 2x dx + \int \text{cosec}^2 3x dx = \frac{\tan 2x}{2} - \frac{\cot 3x}{3} + c$$

Integration of indefinite integrals by the method of substitution

This technique is really an extension of the technique of direct integration. It is often easier to recognise an integral if a substitution can be applied.

The technique of integration by substitution is best learnt through examples.

The formula is:

$$\int (f' \circ g) \times g' = f \circ g$$

but it is from examples that you learn how to use this.

Example

Find $\int \frac{2x}{(2x-1)^2} \; dx$

Let $u = 2x - 1$

Then

$$\frac{du}{dx} = 2$$

$$\therefore dx = \frac{1}{2}du$$

and also $2x = u + 1$

Hence

$$\int \frac{2x}{(2x-1)^2}dx = \int \frac{u+1}{u^2} \cdot \frac{1}{2}du$$

$$= \int \frac{1}{2u} + \frac{1}{2u^2}du$$

$$= \frac{1}{2}\ln|2u| - \frac{1}{2}u^{-1} + c$$

$$= \frac{1}{2}\ln 2(2x-1) - \frac{1}{2}(2x-1)^{-1} + c$$

$$= \ln\sqrt{4x-2} - \frac{1}{2(2x-1)} + c$$

Integration of definite integrals by the method of substitution

Example

Evaluate $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \cdot \cos^4 x dx$

Solution

Method (1): Evaluating in the original variable with no change of limits.

The indefinite integral is

$$\int \sin x \times \cos^4 xdx$$

Making the substitution, $u = \cos x,$ then $du = -\sin x\ dx;$ hence,

$$\int \sin x \times \cos^4 xdx = -\int u^4 du = -\frac{u^5}{5} + c = -\frac{1}{5}\cos^5 x + c$$

Therefore,

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \times \cos^4 x dx = \left[\frac{1}{5}\cos^5 x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{1}{5}\left[\cos^5 x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{1}{40}\sqrt{2}$$

Method (2): Evaluating in the substituted variable with a change of limits.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \times \cos^4 x dx$$

We make the substitution, $u = \cos x$, giving $du = -\sin x \, dx$

but we also replace the limits.

When $x = \frac{3\pi}{4}$, $u = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

When $x = \frac{\pi}{2}$, $u = \cos\frac{\pi}{2} = 0$

hence,

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \sin x \times \cos^4 x dx = -\int_0^{-\frac{1}{\sqrt{2}}} u^4 du = -\left[\frac{u^5}{5}\right]_0^{-\frac{1}{\sqrt{2}}} = -\frac{1}{5}\left(-\frac{1}{4\sqrt{2}} - 0\right) = \frac{1}{40}\sqrt{2}$$

Applications of integration to find areas

Example

Find the area under the curve $y = x^3 - 4x$ between the points (i) 0 and 2; (ii) 2 and 4.

Solution

(i) Area $= \int_a^b f(x)dx$

$$= \int_0^2 x^3 - 4xdx = \left[\frac{1}{4}x^4 - 2x^2\right]_0^2 = (4-8)-(0) = -4$$

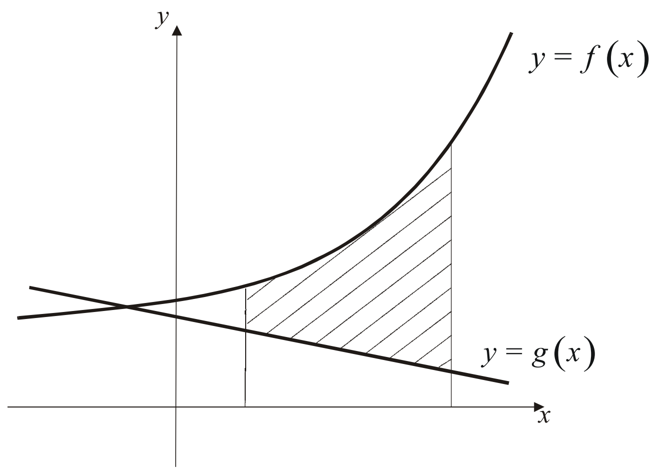
The negative area corresponds to an area under the curve.

(ii) Area $= \int_a^b f(x)dx$

$$= \int_2^4 x^3 - 4x dx = \left[\frac{1}{4}x^4 - 2x^2\right]_2^4 = (64-32)-(4-8) = 32+4 = 36$$

Area Bounded by Two Curves

Suppose we have two functions *f(x)* and *g(x)*.



We are asked to find the area bounded by these two functions with limits *a* and *b*. Then:

Area $= \int_a^b \left[f(x) - g(x)\right].dx$

Example

Find the area bounded by the curves, $y = \sqrt{3x}$ and $y = 2x - 3$.

Answer

Area $= \int_a^b \left[f(x) - g(x)\right].dx$

Here $f(x) = \sqrt{3x}$ and $g(x) = 2x - 3$

We need to find the point of intersection of the two curves:

$\sqrt{3x} = 2x - 3$

$3x = (2x-3)^2 = 4x^2 - 12x + 9$

$4x^2 - 15x + 9 = 0$

$(4x-3)(x-3) = 0$

$x = 3/4$ or $x = 3$

When $x = 3/4$, $y = \sqrt{9/4} = 3/2$

When $x = 3$, $y = 3$

In fact, the $x = 3/4$ solution would correspond to the intersection of the two curves below the x-axis. It shows up as $+3/2$ because we squared the term $\sqrt{3x}$. We discard this solution. Therefore, we require the integral:

$$\text{Area} = \int_0^3 \left[(3x)^{\frac{1}{2}} - (2x-3) \right] .dx$$

$$= \int_0^3 \left[(3x)^{\frac{1}{2}} - 2x + 3 \right] .dx$$

$$= \left[\tfrac{2}{9}(3x)^{\frac{3}{2}} - x^2 + 3x \right]_0^3 = \left(\frac{2 \times 27}{9} - 9 + 9 \right) - (0) = 6$$

Integration by Parts

Integration by parts is the reverse of the process of differentiation of a product.

The product rule for differentiation is:

$$(f \times g)' = f' \times g + f \times g'$$

Rearrangement gives:

$$f \times g' = (f \times g)' - f' \times g$$

We can then integrate both sides to obtain the formula for integration by parts

$$\int f \cdot g' = f \cdot g - \int f' \cdot g$$

In this formula *f* is a function to be differentiated and *g'* is a function to be integrated.

Example

To find $\int x^4 \ln x \, dx$

$f(x) = \ln x \qquad g'(x) = x^4$

$f'(x) = \frac{1}{x} \qquad g(x) = \frac{1}{5}x^5$

The integration by parts formula is

$$\int f \cdot g' = f \cdot g - \int f' \cdot g$$

Substitution into it gives

$$\begin{aligned}\int x^4 \ln x \, dx &= \frac{1}{5}x^5 \times \ln x - \int \frac{1}{x} \times \frac{1}{5} \times x^5 dx \\ &= \frac{1}{5}x^5 \times \ln x - \int \frac{1}{5} \times x^4 dx \\ &= \frac{1}{5}x^5 \times \ln x - \frac{1}{5}\int x^4 dx\end{aligned}$$

And $\int x^4 \, dx = \frac{1}{5}x^5 + c$

So

$$\int x^4 \ln x \, dx = \frac{1}{5}x^5 \times \ln x - \frac{1}{5} \times \frac{1}{5} \times x^5 = \frac{1}{5}x^5 \times \ln x - \frac{1}{25} \times x^5 .$$

Example

Find $\int e^{2x} \cos x.dx$

Solution

$$f(x) = e^{2x} \qquad g'(x) = \cos x$$
$$f'(x) = 2e^{2x} \qquad g(x) = \sin x$$

The integration by parts formula is

$$\int fg' = fg - \int f'g$$

Substitution into it gives

$$\int e^{2x} \cos x \, dx = \text{-}e^{2x} \sin x - \int 2e^{2x} \sin x \, dx$$

We can take the 2 on the right-hand-side outside the integral side

(1) $\int e^{2x} \cos x \, dx = \text{-}e^{2x} \sin x - 2\int e^{2x} \sin x \, dx$

W now need to find $\int e^{2x} \sin x \, dx$.

$$f(x) = e^{2x} \qquad g'(x) = \sin x$$
$$f'(x) = 2e^{2x} \qquad g(x) = \text{-}\cos x$$

Then

$$\int e^{2x} \sin x \, dx = \text{-}e^{2x} \cos x - \int 2e^{2x} \cos x \, dx$$

Substituting for $\int e^{2x} \sin x \, dx$ at (1) gives

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x - 2\left\{\text{-}e^{2x} \cos x + 2\int e^{2x} \cos x \, dx\right\}$$
$$\therefore \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x - 4\int e^{2x} \cos x dx$$

Collecting the terms in $\int e^{2x} \cos x \, dx$ gives

$$5\int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x$$
$$\therefore \int e^{2x} \cos x \, dx = \frac{1}{5}\left(e^{2x} \sin x + 2e^{2x} \cos x\right)$$

The Integral of $1/x$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

Example

$$\int_{-2}^{-1} \frac{1}{x} \, dx = -\int_{-2}^{-1} -\frac{1}{x} \, dx$$
$$= -\left[-\ln(-x)\right]_{-2}^{-1}$$
$$= -\left(-\ln(1) + \ln(2)\right)$$
$$= -\ln(2)$$

The use of $|x|$ shortens this process:

$$\int_{-2}^{-1} \frac{1}{x} \, dx = \left[\ln|x|\right]_{-2}^{-1}$$
$$= \ln(1) - \ln(2)$$
$$= -\ln(2)$$

Integrals of $\cos^2 x$ and $\sin^2 x$

To integrate $\int \sin^2 x dx$

$\cos 2x = 1 - 2\sin^2 x$

$$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\therefore \int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x \right) dx$$

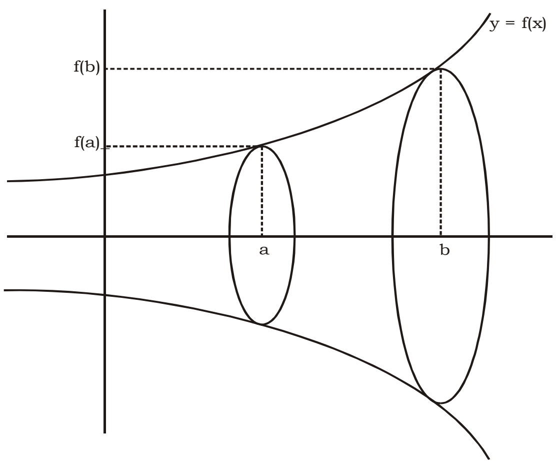
$$= \frac{x}{2} - \frac{1}{4}\sin 2x + c$$

Volume of Revolution about the x-axis

When a function, $y = f(x)$, is rotated about the *x*-axis, the volume generated is given by:

$$V = \pi \int_a^b y^2 dx.$$

Here *a* and *b* are the lower and upper bounds of the volume.



$$V = \int_a^b \pi y^2 dx.$$

Example

Find the volume of revolution when $y = 2x^2$ is rotated about the $x-$axis between $x = 1$ and $x = 3$.

Solution

$$y = 2x^2$$

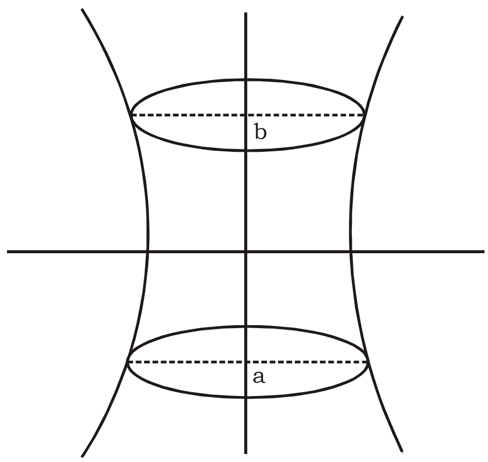
$$V = \int_a^b \pi y^2 dx$$

$$= \int_1^3 \pi \left(2x^2\right)^2 dx = \pi \int_1^3 4x^4 dx = \pi \left[\tfrac{4}{5}x^5\right]_1^3 = \pi\left(\frac{972}{5} - \frac{4}{5}\right) = \frac{968\pi}{5}$$

Volume of Revolution about the y-axis

The volume of revolution about the *y*-axis is:

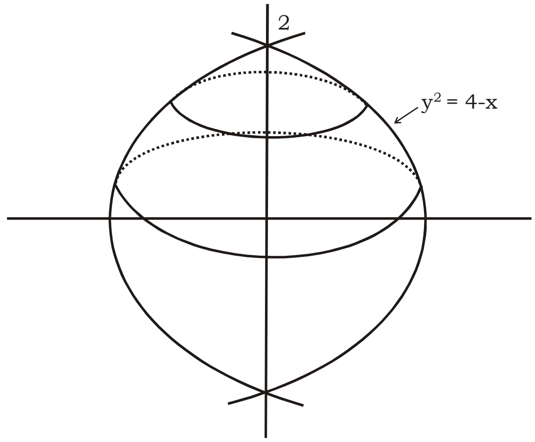
$$V = \pi \int_a^b x^2 dy .$$



The proof is similar to the above proof for the volume of revolution about the *x*-axis.

Example

To find the volume of revolution when $y^2 = 4 - x$ is rotated about the *y*-axis.



To find the limits of the integration:

when $x = 4$ then $y^2 = 4$,
so $y = \pm 2$

$$V = \pi \int_{-2}^{2} x^2 dy$$

$$= \pi \int_{-2}^{2} \left(4 - y^2\right)^2 dy$$

$$= \pi \int_{-2}^{2} 16 - 8y^2 + y^4 dy$$

$$= \pi \left[16y - \frac{8y^3}{3} + \frac{y^5}{5} \right]_{-2}^{2}$$

$$= \left(32 - \frac{64}{3} + \frac{32}{5} \right) - \left(-32 + \frac{64}{3} - \frac{32}{5} \right)$$

$$= 32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} + \frac{32}{5}$$

$$= 64 - \frac{128}{3} + \frac{64}{5}$$

$$= \frac{960 - 640 + 192}{15}$$

$$= \frac{512}{15}$$