Test for the Mean of a Poisson distribution

The test for the mean of a Poisson distribution is best illustrated through a worked example.

Example

(*a*) In the region of a fault line earthquakes can be experiences. A certain town is expected to receive 5 minor tremors registering above 3.0 on the Richter scale in a given year. In one year 10 such tremors were recorded. Model the number of these tremors as a Poisson distribution and test at the 5% level of significance the hypothesis that there has been a change in the average number of tremors per year.

(*b*) Determine the critical region for the test, and state what the actual significance level is; hence, determine the probability of a type I error.

(*c*) Suppose that in fact the mean number of tremors has increased to 7 per year. Find the probability of making a type II error with the test provided by the critical region that you have just identified in part (*b*)

Solution

(*a*)

We have $X \sim Po(5)$

There is no prior expectation that the number of tremors has increased, so the test is a two-tailed test. This means that the critical region at both ends of the distribution should be 2.5%.

Thus, stating the null and alternative hypotheses

$H_0 \quad \mu = 5$

$H_1 \quad \mu \neq 5$

We need to find the probability of 10 or more such tremors under the null hypothesis. To do this we compute the values of the outcomes upto and including $X = 9$, as follows

$$P(X=0) = e^{-5} \times \frac{5^0}{0!} = 0.006738$$

$$P(X=1) = e^{-5} \times \frac{5^1}{1!} = 0.033690$$

$$P(X=2)=e^{-5}\times\frac{5^2}{2!}=0.084224$$

$$P(X=3)=e^{-5}\times\frac{5^3}{3!}=0.140374$$

$$P(X=4)=e^{-5}\times\frac{5^4}{4!}=0.175467$$

$$P(X=5)=e^{-5}\times\frac{5^5}{5!}=0.175467$$

$$P(X=6)=e^{-5}\times\frac{5^6}{6!}=0.146223$$

$$P(X=7)=e^{-5}\times\frac{5^7}{7!}=0.104445$$

$$P(X=8)=e^{-5}\times\frac{5^8}{8!}=0.065278$$

$$P(X=9)=e^{-5}\times\frac{5^9}{9!}=0.036266$$

$$\begin{aligned}P(X\le 9) &= 0.006738+0.033690+0.084224+0.140375+0.175467\\ &\quad +0.175467+0.146223+0.104445+0.065278+0.036266\\ &= 0.968173\end{aligned}$$

Hence

$$P(X\ge 10)=1-P(X\le 9)=1-0.968173=0.031827$$

Since this value is greater than the critical value of 0.025 we do not have evidence to reject the null hypothesis.

Accept H_0, Reject H_1

(b)

The Poisson distribution is a discrete distribution. We require a test that ensures that the probability of rejecting the null hypothesis is less than 5% in total; and for each tail of the distribution, is less than 2.5%. At the lower end of the scale, this is provided by $X = 0$, with probability $P(X=0)=0.006738$. The cumulative probability for $X\le 1$ is given by

$$\begin{aligned}P(X\le 1) &= P(X=0)+P(X=1)\\ &= 0.006738+0.033690\\ &= 0.040428\end{aligned}$$

This is greater than 0.025, so in fact, at the lower tail the critical region is just *X=0* and the significance level is 0.006738.

For the upper end of the tail we need first to calculate the probability that $X = 10$. This is

$$P(X = 10) = e^{-5} \times \frac{5^{10}}{10!} = 0.018133$$

Adding this to the cumulative probability for $P(X \le 9)$, we obtain the cumulative probability for $P(X \le 10)$ as $0.968173 + 0.018133 = 0.986306$

Hence the probability

$$P(X > 10) = 1 - 0.986306 = 0.013694$$

This is less than the desired level of 2.5%, but this is a consequence of any test on a discrete probability distribution.

The critical region in the upper tail is, then, $X > 10$, and the significance level for this tail is 0.013694. We will reject the null hypothesis if

Either $X = 0$ or $X > 10$

The combined significance level is $0.006738 + 0.013694 = 0.020432$ and this is the probability of a type I error, which is the error of rejecting the null hypothesis when in fact it is true.

(*c*)

A type II error is the error of accepting the null hypothesis when in fact it is false. In order to assess a type II error we must have a definite alternative hypothesis, here provided by the assertion that the mean is in fact 7 tremors in a given year. Thus, we seek the probability

$$P(\text{accept } H_0 | \mu = 7)$$

We accept the null hypothesis if the test result is $1 \le X \le 10$ so we need to find the probability of *X* lying in this region under the hypothesis that $X \sim Po(7)$

We begin as usual by computing the probabilities

$$P(X = 1) = e^{-7} \times \frac{7^1}{1!} = 0.006383$$

$$P(X=2)=e^{-7}\times\frac{7^2}{2!}=0.022341$$
$$P(X=3)=e^{-7}\times\frac{7^3}{3!}=0.052129$$
$$P(X=4)=e^{-7}\times\frac{7^4}{4!}=0.091226$$
$$P(X=5)=e^{-7}\times\frac{7^5}{5!}=0.127717$$
$$P(X=6)=e^{-7}\times\frac{7^6}{6!}=0.149003$$
$$P(X=7)=e^{-7}\times\frac{7^7}{7!}=0.149003$$
$$P(X=8)=e^{-7}\times\frac{7^8}{8!}=0.130377$$
$$P(X=9)=e^{-7}\times\frac{7^9}{9!}=0.101405$$
$$P(X=10)=e^{-7}\times\frac{7^{10}}{10!}=0.070983$$

Hence the probability of the test result falling inside this acceptance region, under the hypothesis that the mean is in fact 7, is given by

$$\begin{aligned}P &= 0.006383+0.022341+0.052129+0.091226+0.127717\\ &\qquad +0.149003+0.149003+0.130377+0.101405\\ &= 0.829584\\ &= 83.0\%\,(3.S.F)\end{aligned}$$

This error is very high. In other words, with the significance level set in the way determined in part (*b*), the test is very likely not to record a change in the mean.

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