Testing the Sample Mean

Prerequisites

You should be familiar with (1) determining a confidence interval for a sample mean drawn from a population of known or unknown variance and hence the biased and unbiased estimator of the population variance; (2) the central limit theorem.

Testing the sample mean with known variance

Statistical quality control
Throughout business there is a need to check whether industrial processes are functioning correctly. For example, a machine that produces components does so to a certain specification that is capable of being described by a random variable *X*. The machine is calibrated to produce the components to a certain value (the mean, μ) and to a certain tolerance (the variance, σ^2). The question is whether the calibration is correct. To determine whether this is so, rather than examine every component, which is costly, a sample is regularly drawn. If the sample mean is significantly different from the intended mean then it is concluded that the machine needs recalibration. This is part of *statistical quality control*, and an important application of statistics throughout the world.

Essentially, in this case, we are comparing a sample mean $\bar{X}$ with an expected population mean μ and asking the question – could the sample be drawn from the population? This is a hypothesis test of the difference between the sample mean and the population mean. This question can only be answered in terms of probability – in other words, to a certain level of probability it is either likely or unlikely that the sample has been drawn from the population. So the hypothesis test, as ever, involves a level of significance, which is the predetermined level of probability that we shall use to determine whether to accept or reject the null hypothesis that the sample mean is the same as the population mean.

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Testing the mean with known variance

Let us suppose that a random variable X is normally distributed with mean μ and **known** population variance σ^2. A sample is drawn with sample mean $\bar{X}$ and sample variance S^2. We wish to determine whether the sample could be drawn from this population. Suppose initially that the population is believed to be normally distributed, so that

$$X \sim N\left(\mu,\ \sigma^2\right)$$

then by the central limit theorem the sample mean is also normally distributed.

$$\bar{X} \sim N\left(\mu,\ \frac{\sigma^2}{n}\right)$$

Even if the parent population is not normally distributed, then by the central limit theorem we may still infer that the sample mean is normally distributed as above, provided that the sample size is large enough $n > 30$. Thus in either case we can use the usual statistic

$$z_{test} = \frac{\left|\bar{X} - \mu\right|}{\sigma/\sqrt{n}}$$

to evaluate the hypothesis. This z-score is compared with a critical value found from tables in accordance with (*a*) the significance level of the test, and (*b*) whether the test is one or two-tailed.

Example (1)

The volumes of cans produced by a certain machine are normally distributed with mean volume 502 cm^3 and standard deviation 14 cm^3. Following a service a sample of 80 cans is taken and it is found that the sample mean is now 497cm^3. Is there any evidence at the 5% level that the service has caused a change in the mean volume of the cans manufactured by the machine?

Solution

This is a two-tailed test since the result of the service could alter the mean volume in either direction. Hence the hypotheses are

H_0 $\mu = 502$

H_1 $\mu \neq 502$ two tailed, significance level $\alpha = 0.05$

$$z_{test} = \frac{\left|\bar{X} - \mu\right|}{\sigma/\sqrt{n}} = \frac{\left|497 - 502\right|}{14/\sqrt{80}} = 3.194 \quad \text{(3 d.p.)}$$

$$z_{critical} = P\left(Z > 0.975\right) = 1.960$$

$$z_{test} > z_{critical}$$

Reject H_0, accept H_1

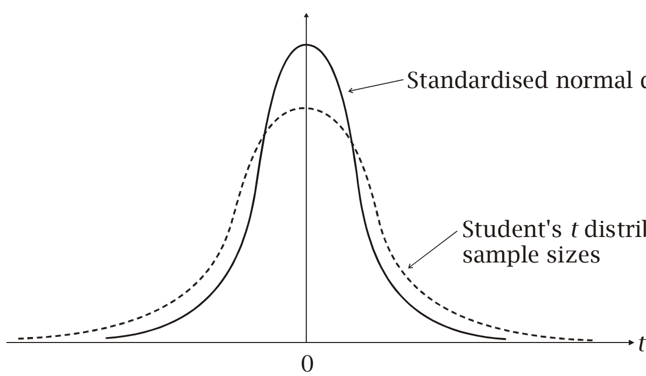
The service has altered the mean volume of the cans.

Testing the sample mean with unknown variance

Example (1) deals with the case where the variance is known. Furthermore, it applies where either the population is known to be normally distributed or the sample size is $(n > 30)$. However, this raises the question of what do we do when the variance is not known and the sample size is small $(n \leq 30)$. Because the sample size is small we must assume that the parent population is normally distributed. To answer this question, in the first place we determine the same statistic

$$\text{test statistic} = \frac{\left|\bar{X} - \mu\right|}{\sigma / \sqrt{n}}$$

but substitute the unbiased estimate of the population variance derived from the sample itself. However, a complication arises. There is a possibility that the unbiased sample variance may not be accurate. Hence, it is advisable to use a probability distribution that accounts for this. Intuitively, it would be a modification of the standardised normal distribution that makes adjustments for the increased possibility that the "tails" of the distribution would be fatter because of the additional source of random error. The distribution that does this is called the *Student's t distribution*. It was discovered by William Goset a statistician working for the company Guiness. Guiness did not permit publication of work connected to the company, so Goset decided to publish under the pseudonym of "Student". That is why it is known as "Student's *t*". The following diagram gives an impression of how the *t*-distribution for low sample values compares with the *z*- distribution looks something like the following.



With this adjustment, the resultant *t*-test is similar to the *z*-test, and the test is conducted in the same way, with the tables for Student's-*t* used in place of those for the standardised normal variable in order to determine the critical value. The graphs comparing Student's *t* to the standardised *z* indicates a further complication with the application of this test. As the sample size increases the estimation of the population variance improves. In other words, as *n* increases Student's *t* converges on the standardised normal distribution.

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$t \to z$ as $n \to \infty$

Thus, the difference between the *t*-test and the *z*-test diminishes, as *n* gets larger. What this means is that the critical values of the *t*-test depend on the sample size of *n*. Furthermore, since the *t* distribution is used primarily in the context of hypothesis testing, full *t*-tables for each value of *n* are not usually given. Instead, critical values of the *t*-test for each significance or probability level are quoted. Finally, the critical values are not cited for the sample size, *n*, themselves but for the number of *degrees of freedom*, ν, where

$\nu = n - 1$ degrees of freedom (ν) = sample size $(n) - 1$

The table for the Student's *t*-distribution commences as follows.

ν	$t_{0.55}$	$t_{0.60}$	$t_{0.70}$	$t_{0.75}$	$t_{0.80}$	$t_{0.90}$	$t_{0.95}$	$t_{0.975}$	$t_{0.99}$	$t_{0.995}$
1	0.158	0.325	0.727	1.000	1.376	3.08	6.31	12.71	31.82	63.66
2	0.142	0.289	0.167	0.816	1.061	1.89	2.92	4.30	6.96	9.92
3	0.137	0.277	0.584	0.765	0.978	1.64	2.35	3.18	4.54	5.84
4	0.134	0.271	0.569	0.741	0.941	1.53	2.13	2.78	3.75	4.60
5	0.132	0.267	0.559	0.727	0.920	1.48	2.02	2.57	3.36	4.03
6	0.131	0.265	0.553	0.718	0.906	1.44	1.94	2.45	3.14	3.71
7	0.130	0.263	0.549	0.711	0.896	1.42	1.90	2.36	3.00	3.50
8	0.130	0.262	0.546	0.706	0.889	1.40	1.86	2.31	2.90	3.36
9	0.129	0.261	0.543	0.703	0.883	1.38	1.83	2.26	2.82	3.25
10	0.129	0.260	0.542	0.700	0.879	1.37	1.81	2.23	2.76	3.17

Example (2)

A certain machine process is said to yield components of average length 10.30 cm. A random sample of eight components was taken and their lengths recorded as follows.

9.85 10.20 10.25 9.90 10.35 10.15 10.20 10.05

Calculate the sample mean and unbiased population variance. Use a *t*-test, with significance level 1% to test the claim that the population mean is 10.30 cm. State any assumptions.

Solution

The hypotheses are

H_0 $\mu = 10.30$

H_1 $\mu \neq 10.30$ two-tailed, $\alpha = 1\%$

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The test statistic is computed as follows.

$$\bar{X} = \frac{\sum X}{n} = 10.11875$$

$$S^2 = \frac{\sum X^2}{N} - \left(\bar{X}^2\right) = 0.02621...$$

$$\hat{\sigma}^2 = S^2 = \frac{n}{n-1}S^2 = \frac{8}{7} \times 0.02621... = 0.02995...$$

$$\begin{aligned} \mathrm{t}_{\text{test}} &= \frac{\left|\bar{X} - \mu\right|}{\hat{\sigma}/\sqrt{n}} \\ &= \frac{\left|110.11875 - 10.30\right|}{\sqrt{\frac{0.02995...}{8}}} \\ &= \frac{0.18125}{0.06119...} \\ &= 2.962 \qquad \text{(3 d.p.)} \end{aligned}$$

ν = degrees of freedom = $n - 1 = 8 - 1 = 7$

$t_{\text{critical}}\left(\nu = 7, p = 0.995\right) = 3.50$ (two-tailed test)

$t_{test} < t_{critical}$

Therefore, reject H_1, accept H_0.

The average length of the components is 10.30 cm.

We assume that average length of a component is normally distributed.

For large samples

The discussion of the convergence of Student's t-distribution the standardised normal distribution indicates that when the sample size is sufficient large $(n > 30)$ the difference between t and z is negligible. Hence, when $n > 30$ we revert to the z-test and do not use the t-distribution.

Summary of Student's *t*

Given a random variable X with true mean μ, expected mean μ_0 and unknown variance σ^2 and a sample of size $n < 30$ with sample mean $\bar{X}$ and sample variance S^2, to test the null hypothesis

H_0 $\quad \mu = \mu_0$

against the alternative hypothesis

H_1 $\quad \mu \neq \mu_0$ two-tailed

or alternatively

H_1 $\quad \mu > \mu_0$ or $\mu < \mu_0$ one-tailed

at a given significance level Then the test statistic is given by

$$t_{test} = \frac{\left|\bar{X} - \mu\right|}{\hat{\sigma}/\sqrt{n}}$$

where $\hat{\sigma}^2$ is the unbiased estimator of the population mean given by

$$\hat{\sigma}^2 = s^2 = \frac{n}{n-1}S^2$$

The critical value is drawn from Student's *t*-distribution, where the number of degrees of freedom is given by

$\nu = n - 1$ degrees of freedom $(\nu) =$ sample size $(n) - 1$

The assumption of this test is that the population is normally distributed $X \sim N\left(\mu,\ \sigma^2\right)$.

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