The algebra of complex numbers

Prerequisites: addition and subtraction of complex numbers

You should already be familiar with the idea of complex numbers, their representation in the Argand plane in Cartesian, polar and triognometric form, and how to add them.

Cartesian form

$z = x + iy = (x, y)$

Polar form

$z = [r, \theta] = [|z|, \arg z]$

Trigonometric form

$z = r(\cos\theta + i\sin\theta)$ where $z = [r, \theta]$

Example (1)

$$z = \left[2, -\frac{\pi}{4}\right]$$

Find the Cartesian and trigonometric forms of *z*.

Solution

$$x = r\cos\theta = 2\cos\left(-\frac{\pi}{4}\right) = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$y = r\sin\theta = 2\sin\left(-\frac{\pi}{4}\right) = 2 \times -\frac{1}{\sqrt{2}} = -\sqrt{2}$$

Cartesian form

$$z = (x, y) = \left(\sqrt{2}, -\sqrt{2}\right) = \sqrt{2} - i\sqrt{2}$$

Trigonometric form

$$z = 2\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right)$$

To add complex numbers, add the components in Cartesian form

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

Example (2)

$$z_1 = 3 + 4i,\ \ z_2 = -1 - i$$
$$z_1 + z_2 = (3 + 4i) + (-1 - i) = (3 - 1) + i(4 - 1) = 2 + 3i$$

Subtraction follows the obvious rule:

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

Example (3)

$$z_1 = 3 + 4i,\ \ z_2 = -1 - i$$
$$z_1 - z_2 = (3 + 4i) - (-1 - i) = (3 + 1) + i(4 + 1) = 4 + 5i$$

Multiplication in Cartesian coordinates

When complex numbers are given in Cartesian form, the rule for multiplying them is

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

That is, expand the brackets in the usual way and use the rule that $i^2 = -1$

Example (4)

$$z_1 = 3 + 4i,\ \ z_2 = -2 + i$$
$$z_1 z_2 = (3 + 4i)(-2 + i) = -6 + 4i^2 + 3i - 8i = -10 - 5i$$

Division in Cartesian coordinates is slightly more involved and is described below.

Multiplication in polar coordinates

if $z_1 = [r_1, \theta_1]$, and $z_2 = [r_2, \theta_2]$

$$\begin{aligned} z_1 z_2 &= [r_1, \theta_1][r_2, \theta_2] \\ &= [r_1 r_2, \theta_1 + \theta_2] \end{aligned}$$

The rule is: multiply the moduli and add the arguments. This may seem initially more difficult, but is in fact easier than multiplication in the Cartesian form

Example (5)

if $z_1 = \left[2, \frac{\pi}{4}\right], \; z_2 = \left[3, \frac{\pi}{6}\right]$

Find $z_1 z_2$ in polar form

Solution

$$\begin{aligned} z_1 z_2 &= \left[2 \times 3, \frac{\pi}{4} + \frac{\pi}{6}\right] \\ &= \left[6, \frac{5\pi}{12}\right] \end{aligned}$$

Division in polar coordinates

Division in polar coordinates is the inverse of the process of multiplication in polar coordinates: divide the moduli and subtract the arguments.

$$\frac{z_1}{z_2} = \left[\frac{r_1}{r_2}, \theta_1 - \theta_2\right]$$

Example (6)

if $z_1 = \left[2, \frac{\pi}{4}\right], \; z_2 = \left[3, \frac{\pi}{6}\right]$

Find $\frac{z_1}{z_2}$ in polar form

Solution

$$\frac{z_1}{z_2} = \left[2 \div 3, \frac{\pi}{4} - \frac{\pi}{6}\right]$$
$$= \left[\frac{2}{3}, \frac{\pi}{12}\right]$$

Proof of the formula for multiplication in polar coordinates

We should prove the formula for multiplication in polar coordinates. To prove

$z_1 = [r_1, \theta_1]$, and $z_2 = [r_2, \theta_2]$

Then

$$\begin{aligned} z_1 z_2 &= [r_1, \theta_1][r_2, \theta_2] \\ &= [r_1 r_2, \theta_1 + \theta_2] \end{aligned}$$

Proof

$$\begin{aligned} z_1 z_2 &= [r_1, \theta_1][r_2, \theta_2] \\ &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \\ &= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\ &= [r_1 r_2, \theta_1 + \theta_2] \end{aligned}$$

This proof is a useful revision of various trigonometric identities. Proof of the division rule proceeds in the same way.

Complex conjugate and division in Cartesian coordinates

To divide directly in Cartesian coordinates we require the notion of a complex conjugate.

If $z = x + iy$ then the complex conjugate is $\bar{z} = x - iy$.

Sometimes the symbol z^* is used to denote the complex conjugate.

Example (7)

If $z = 2 + 3i$ then $\bar{z} = 2 - 3i$

The point of a complex conjugate is that when it is multiplied by the original complex number, the resulting number has no imaginary part.

Example (8)

If $z = 2+3i$ find $z.\bar{z}$

$$\begin{aligned} z.\bar{z} &= (2+3i)(2-3i) \\ &= 4-9i^2+6i-6i \\ &= 13 \end{aligned}$$

To prove this as a general result, true for all complex numbers

Let $z = x+iy$ then

$$\begin{aligned} z.\bar{z} &= (x+iy)(x-iy) \\ &= x^2-i^2y^2+ixy-ixy \\ &= x^2+y^2 \end{aligned}$$

Division in Cartesian coordinates is achieved by multiplying top and bottom of the fraction by the complex conjugate of the bottom number.

$$\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}}$$

This eliminates the imaginary part from the bottom of the fraction.

Example (9)

$z_1 = 2+3i, \quad z_2 = -3+4i$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(2+3i)}{(-3+4i)} \times \frac{(-3-4i)}{(-3-4i)} \\ &= \frac{6-17i}{25} \end{aligned}$$

Dividing complex numbers in Cartesian form by conversion to polar form

One strategy for dividing complex numbers in Cartesian form is to convert them to polar form, divide, and then convert back.

Example (10)

$z_1 = 2+3i, \;\; z_2 = -3+4i$

Find $\dfrac{z_1}{z_2}$

Solution

$z_1 = \left[\sqrt{13}, \tan^{-1}\left(\frac{3}{2}\right)\right] = \left[\sqrt{13},\; 0.982...\right]$

$z_2 = \left[5, \tan^{-1}\left(\frac{4}{-3}\right)\right] = \left[5,\; 2.214...\right]$

$\dfrac{z_1}{z_2} = \left[\dfrac{\sqrt{13}}{5},\; 0.982.. - 2.214...\right]$

$= \left[0.721,\; -1.23\right]$ (3.D.P.)

Solving equations involving complex numbers

One may be asked to solve equations of the form

Example (11)

Solve $\quad \dfrac{z}{z+2} = 2+3i$

To solve this problem one has to substitute $z = x + iy$ at some stage and then equate the real and imaginary parts. The technique is best demonstrated by a worked example

Solution to problem (11)

$\dfrac{z}{z+2} = 2+3i$

$z = (2+3i)(z+2)$ [Remove the awkward fraction by cross-multiplying]

$z = 2z + 4 + 3iz + 6i$ [Open the bracket]

$-z - 4 = 3iz + 6i$

$-(x+iy) - 4 = 3i(x+iy) + 6i$ [Now substitute $z = x+iy$, the crucial step]

$-x - iy - 4 = 3ix - 3y + 6i$

$3y - x + i(-y - 3x) = 4 + 6i$

$\begin{cases} 3y - x = 4 \\ -y - 3x = 6 \end{cases}$ [Equate real and imaginary parts to obtain two simultaneous equatons]

Solving these simultaneously we obtain

$x = -\dfrac{11}{5} \qquad y = \dfrac{3}{5}$