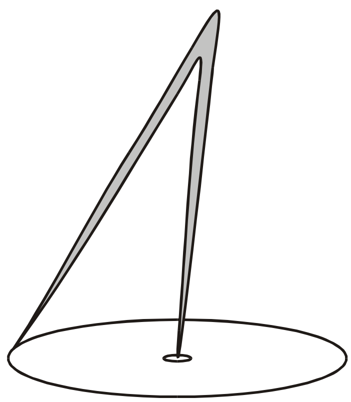
The Cartesian Equation of the Circle

The locus of a circle

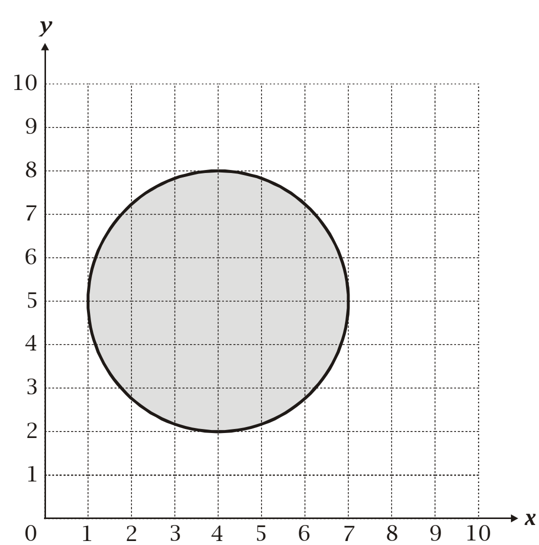
To draw a circle take a pair of compasses, place the foot at the *centre* of the circle, set the distance between the foot and the pen to be the *radius* and draw the circle.



A circle is defined by its *centre* and its *radius*. The path the circle takes is called a *locus*. This locus always remains the same distance away from the centre.

Example (1)

Determine the *centre* and *radius* of this circle.



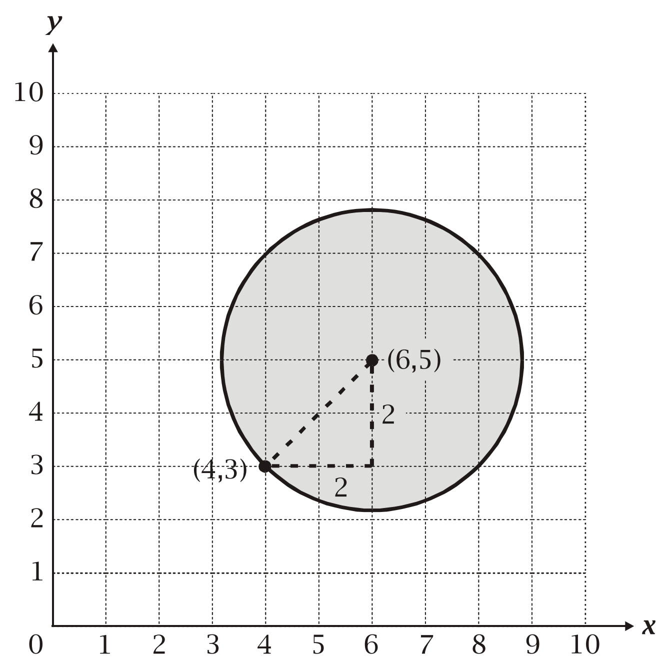
Solution

The *centre* is at $x = 4$, $y = 5$ or $(4,5)$ and the radius 3 units.

Example (2)

A circle has centre $(6,5)$ and goes through the point $(4,3)$. What is its radius?

Solution



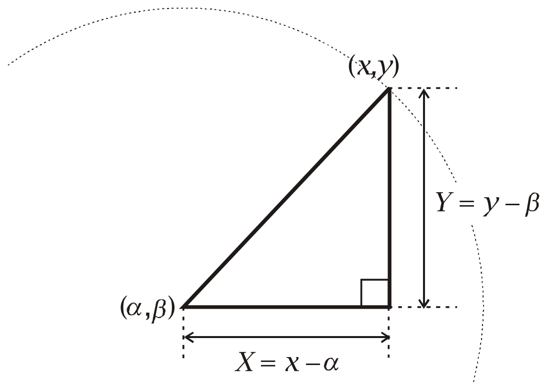
To find the radius we used *Pythagoras's Theorem* – that is in any right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

$r^2 = 2^2 + 2^2 = 4 + 4 = 8$

$r = 2\sqrt{2}$

The Cartesian equation of the circle

We seek an *algebraic expression* for the equation of the circle in general. Let (α, β) represent the *centre* of the circle and let r be its radius. Suppose we have a circle with centre (α, β) and this circle goes through the point (x, y).



In this diagram we have marked the lengths of the right-angled triangle formed on the radius by x and Y. We have

$$X = x - \alpha$$
$$Y = y - \beta$$

From Pythagoras's Theorem

$$r^2 = X^2 + Y^2$$
$$= (x-\alpha)^2 + (y-\beta)^2$$

Hence

$$r = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$$

This gives us a *general relationship* between the centre of a circle (α, β) its radius, r, and *any* point on the circle (its locus) given by (x, y). The equation

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

is called the *Cartesian form* of the equation of the circle. This is the equation of the circle in general – but if the circle is centred on the origin, that is, the point with coordinates $(0,0)$, then $\alpha = 0,\ \beta = 0$ and the equation is *simplified* to

$$x^2 + y^2 = r^2$$

Example (3)

Find the Cartesian form of the equation of the circle with centre (–4, 3) and radius 5

Solution

The general equation of the circle is

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

Here the centre is $(\alpha,\beta) = (-4,3)$ and the radius is $r = 5$. So on substituting

$$(x-(-4))^2 + (y-3)^2 = 5^2$$
$$(x+4)^2 + (y-3)^2 = 5^2$$

Example (4)

Find the centre and radius of the circle whose Cartesian equation is

$$x^2 - 6x + y^2 + 12y + 29 = 0$$

Solution

In order to solve this equation we must put

$$x^2 - 6x + y^2 + 12y + 29 = 0$$

into the form

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

To do this we use the technique of *completing the square*. We turn the $x^2 - 6x$ and $y^2 + 12y$ into completed squares, for $(x-\alpha)^2$ and $(y-\beta)^2$ are squares. To complete the square on

$$x^2 - 6x + y^2 + 12y + 29 = 0$$

for both the x and y terms, take the coefficient of each of the x and y terms in turn, halve it, square it, add it and subtract it.

$$(x^2 - 6x) + (y^2 + 12y) + 29 = 0$$
$$(x^2 - 6x + 3^2 - 3^2) + (y^2 + 12y + 6^2 - 6^2) + 29 = 0$$
$$(x-3)^2 - 9 + (y+6)^2 - 36 + 29 = 0$$
$$(x-3)^2 + (y+6)^2 = 16$$
$$(x-3)^2 + (y+6)^2 = 4^2$$

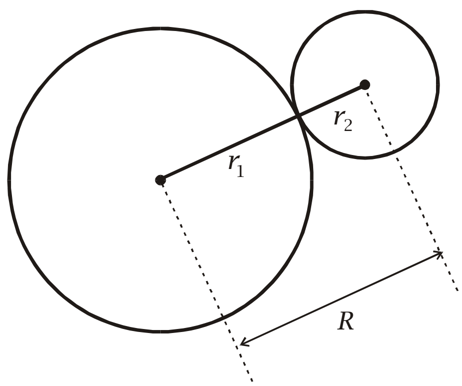
This is now in the form

$$(x-\alpha)^2 + (y-\beta)^2 = r^2$$

From which we can immediately read the position of the centre as $(3,-6)$ and the radius as 4.

Problem solving with the equation of the circle

If two circles C_1 and C_2 with radii r_1 and r_2 respectively just touch, then the sum of the radii must be equal to the distance between their two centres.



In the diagram the distance between the centres of the two circles *R*. If we show that *R* is such that $R = r_1 + r_2$ we prove that the two circles touch.

Example (5)

Prove that the circles C_1 and C_2 given by $(x-8)^2 + (y-6)^2 = 5$ and $x^2 - 4x + y^2 - 6y = 7$ respectively touch.

Solution

The centre of C_1 is $(8,6)$ and its radius is $r_1 = \sqrt{5}$.

We must place the equation for C_2 into completed square form.

$$x^2 - 4x + y^2 - 6y = 7$$
$$x^2 - 4x + 2^2 - 2^2 + y^2 - 6y + 3^2 - 3^2 = 7$$
$$(x-2)^2 + (y-3)^2 = 7 + 4 + 9$$
$$(x-2)^2 + (y-3)^2 = 20$$

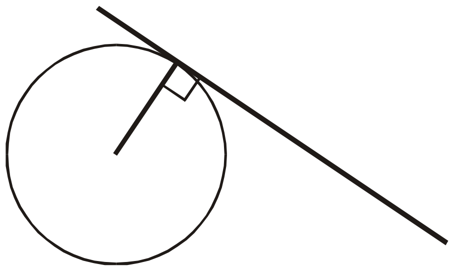
Therefore the centre of C_2 is at $(2,3)$ and its radius is $r_2 = 2\sqrt{5}$.

The distance from C_1 to C_2 is

$$R = \sqrt{(8-2)^2 + (6-3)^2}$$
$$= \sqrt{6^2 + 3^2}$$
$$= \sqrt{45} = 3\sqrt{3}$$

Therefore $R = r_1 + r_2$ and the two circles touch.

A *tangent* to any curve is a line that just touches it. If a line l is tangent to a circle C at a point p, then the angle made by that line l and the radius joining the centre of C to p is a right angle.



Example (6)

Prove that the line l given by

$$y + x = 2 + \sqrt{2}$$

is tangent to the circle C given by

$$x^2 + \left(y - \sqrt{2}\right)^2 = 2$$

and find the equation of the radius of C that meets this tangent giving your answer in the form $ay + bx + c = 0$ where *a, b, c* are numbers to be determined.

Solution

If l is tangent to C then their two equations will have only one solution – representing the single point of intersection of the line and the circle. From

$$y + x = 2 + \sqrt{2}$$

by rearrangement

$$y = \left(2 + \sqrt{2}\right) - x$$

Substituting into $x^2 + \left(y - \sqrt{2}\right)^2 = 2$

$$x^2 + \left(\left(2 + \sqrt{2}\right) - x - \sqrt{2}\right)^2 = 2$$
$$x^2 + (2 - x)^2 = 2$$
$$x^2 + 4 - 4x + x^2 = 2$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$
$$x = 1 \text{ (twice)}$$
$$y = 1 + \sqrt{2}$$

There is only one point of intersection, so l must touch C.

The gradient of the radius at the point of intersection makes a right angle with *l*. From the equation of *l*, $y = (2+\sqrt{2}) - x$ we see that its gradient is –1, so the gradient of the radius is 1, using the principle that the gradients m_1 and m_2 of two perpendicular lines are such that $m_1 m_2 = -1$. The radius passes through the point $(1, 1+\sqrt{2})$ so its equation is

$$\frac{y-(1+\sqrt{2})}{x-1} = 1$$
$$y - 1 - \sqrt{2} = x - 1$$
$$y - x - \sqrt{2} = 0$$