The equation of the straight line

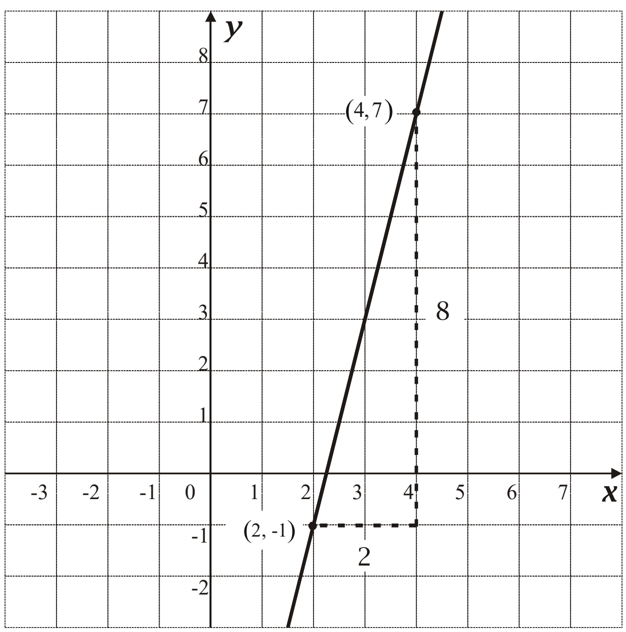
Prerequisites

You should be familiar with the process of finding a gradient of a line joining two points. Let us consolidate this.

Example (1)

Find the gradient of the line joining the points $(2,\ -1)$ and $(4,7)$

Solution



$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$= \frac{7-(-1)}{4-2}$$

$$= \frac{8}{2}$$

$$= 4$$

The gradient is the "rise" over the "step" of the line, shown in the above working as

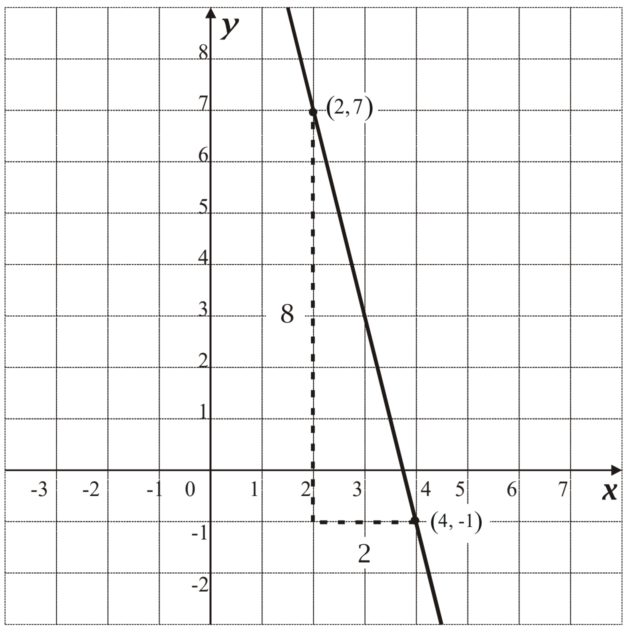
$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

The direction in which the line is sloping is important. The line in this current question is upwards sloping and has a positive gradient, $m > 0$. A downwards-sloping line has a negative gradient.

Example (2)

Find the gradient of the line joining the points $(2, 7)$ and $(4, -1)$

Solution



$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= \frac{7-(-1)}{2-4}$$
$$= \frac{8}{-2}$$
$$= -4$$

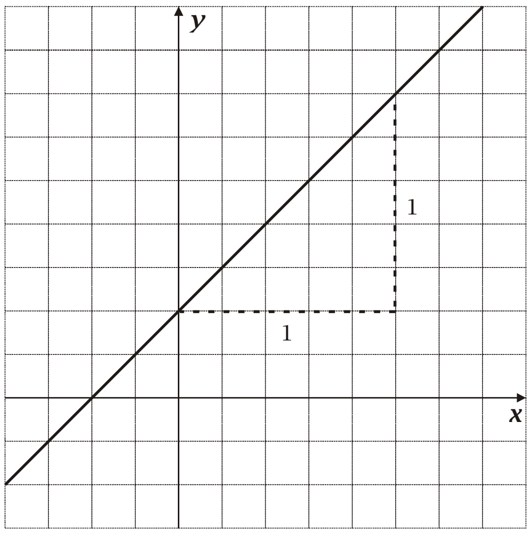
Substituting the coordinates consistently into the formula results in a negative value for the gradient, which is to be expected because the direction of the line is different.

Gradients

The effect of the gradient on the slope of the graph is illustrated by the following cases.

(1) $m = 1$

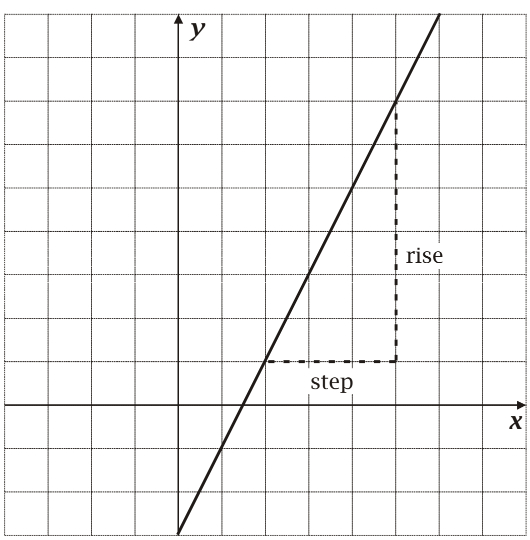
In this case the gradient is positive and equal to 1. This means that the line is upwards sloping and has a 1 in 1 gradient – the rise is equal to the step.



Example of a line with a gradient $m = 1$

(**2**) $m > 1$

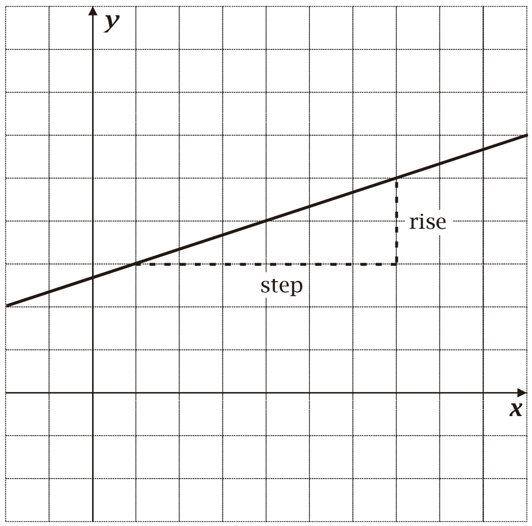
The gradient is positive and greater than 1. The line is upwards sloping and the rise is greater than the step.



Example of a line with gradient $m > 1$

(**3**) $0 < m < 1$

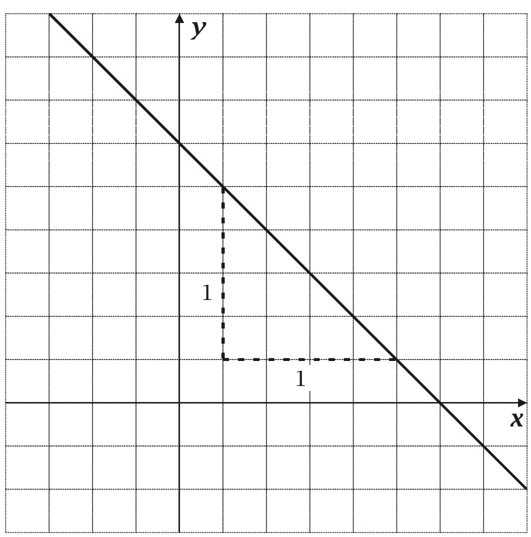
The gradient is positive but it is less than 1. The line is upwards sloping, but it is shallow. The rise is less than the step.



Example of a line with gradient $0 < m < 1$

(**4**) $m = -1$

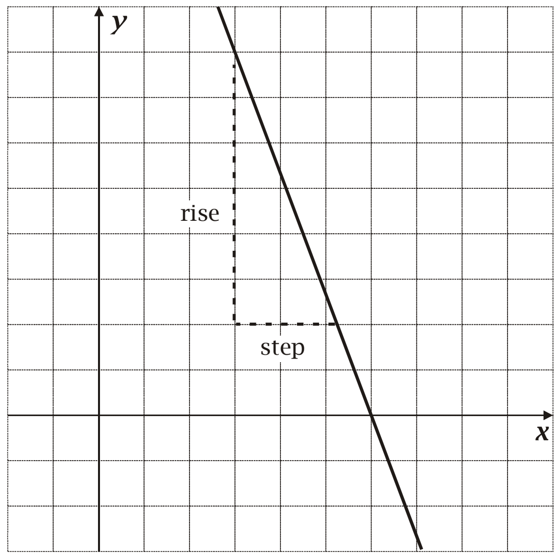
The gradient is negative and is equal to 1. The line is downwards sloping and the rise is equal to the step.



Example of a line with gradient $m = -1$

(**5**) $m < -1$

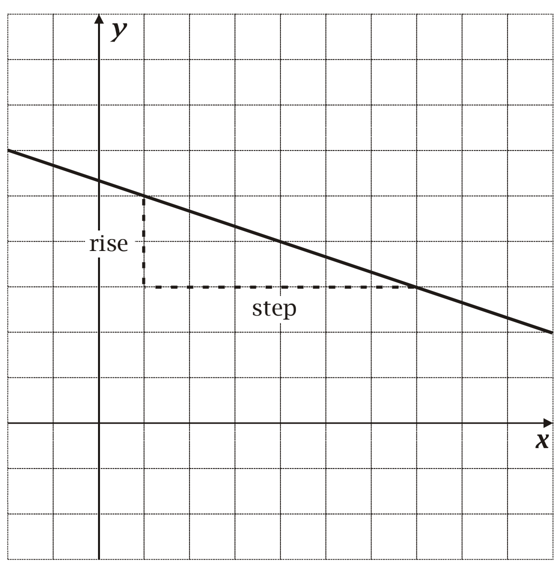
The gradient is negative but its size is greater than 1. It is a steep downwards-sloping line with the rise larger than the step.



Example of a line with gradient $m < -1$

(**6**) $-1 < m < 0$

The gradient is negative but its size is less than 1. It is a shallow downwards-sloping line with the rise less than the step.



Example of a line with gradient $-1 < m < 0$

Conclusion

The effect of the gradient on a line is to alter its orientation, determining whether it is forwards or backwards sloping, and whether it is a steep line or a shallow one. We call the gradient m a *parameter*. The conclusion is that the orientation of a line depends on the parameter m.

The intercept

There is another parameter that determines the position of a line. It is called the *intercept*.

Example (3)

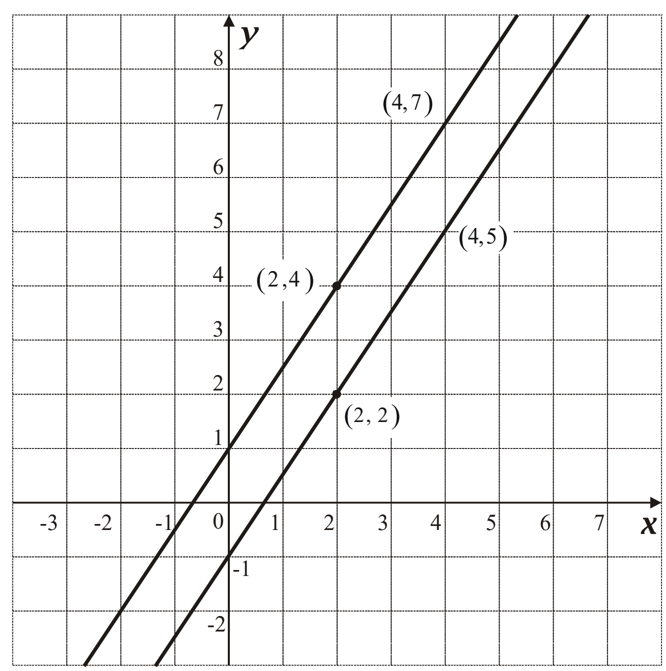
Sketch on a single diagram the two lines joining

(i) $(2,2)$ to $(4,5)$

(ii) $(2,4)$ to $(4,7)$

In each case extend the line so that it cuts the y-axis. Find the gradient of each line and comment on (*a*) what is similar about them, and (*b*) what is dissimilar.

Solution



The gradient of the first line is $m = \frac{5-2}{4-2} = \frac{3}{2}$.

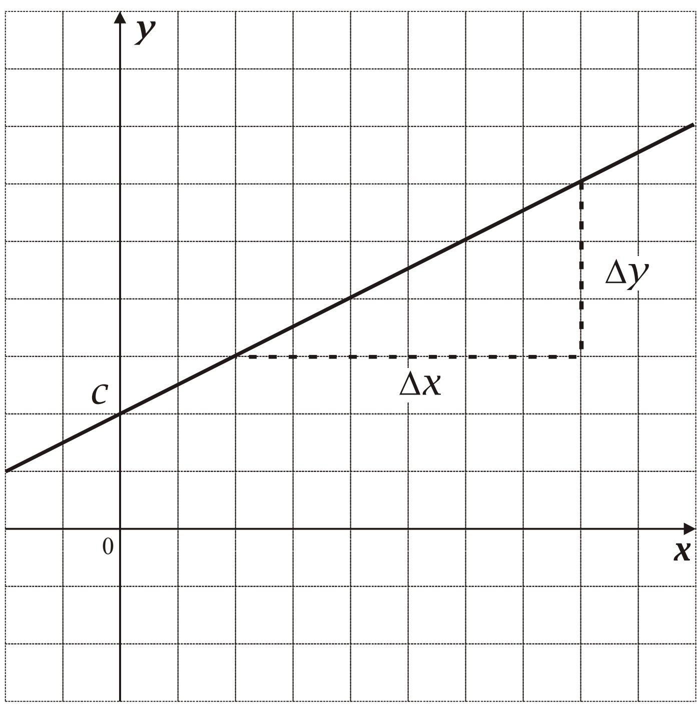
The gradient of the second line is $m = \frac{7-4}{4-2} = \frac{3}{2}$.

From the graph the first line intercepts the y-axis at the point $(0,1)$. The second line intercepts the y-axis at the point $(0,-1)$. As the two lines have the same gradient they are parallel; however, they differ in that they intercept the y-axis at different points.

The intercept of a line with the y-axis is the second parameter that determines the position of a line. A line is uniquely determined by its *gradient* and its *intercept*. Any two lines with the same gradient and intercept must be the same line.

The equation of the straight line

The general *equation of a straight line* is $y = mx + c$ where m is the gradient of the line and c is its intercept on the y-axis.



$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

Example (4)

Find the equation of the straight line that passes through the points $(2, -1)$ and $(4,7)$.

Solution

In example (1) we saw that the gradient of the line passing through the points $(2, -1)$ and $(4,7)$ is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{8}{2} = 4$$

The equation of the straight line is

$y = mx + c$

Substituting for m we obtain

$y = 4x + c$

This equation is satisfied by every point on the line. Hence, substituting the coordinates of either point given will enable us to find the intercept c. Substituting $x = 2,\ y = -1$ gives

$-1 = 4 \times 2 + c$

$c = -9$

Thus on substituting $y = 4x - 9$ into $y = mx + c$ we obtain as solution

$y = 4x - 9$

Other forms of the equation of the straight line

The equation of the straight line can take different forms. In the previous example the solution was $y = 4x - 9$ which takes the form $y = mx + c$. The equation can be expressed in the form $ay + bx + c = 0$.

Example (5)

Express the equation $y = 4x - 9$ in the form $ay + bx + c = 0$.

Solution

Rearrangement gives

$y = 4x - 9$

$y - 4x = -9$

$y - 4x + 9 = 0$

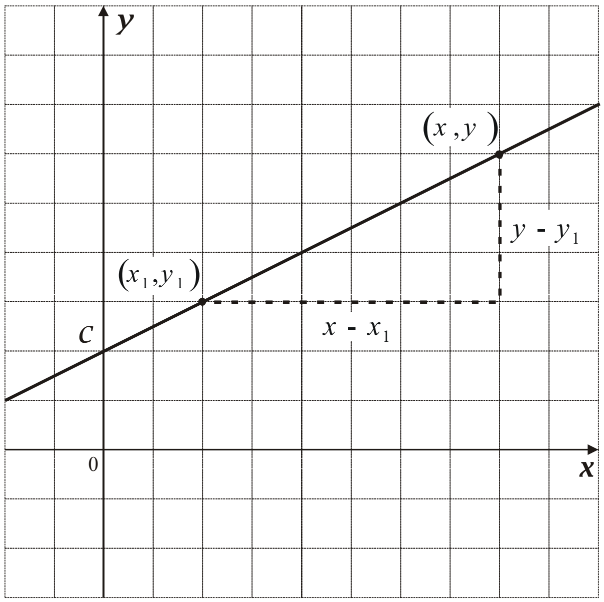
A third form may be derived from the equation for the gradient, which is

$$m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

If (x, y) is a general point on the line and (x_1, y_1) is a specific point, then this formula may be written

$m = \dfrac{y - y_1}{x - x_1}$

The following diagram should make this clear.



Rearrangement of this equation gives a third form in which the equation of the line $y - y_1 = m(x - x_1)$.

Example (6)

Find the equation of the line with gradient $m = -\dfrac{1}{2}$ that passes through the point $(1,2)$.

Find also its intercept with the y-axis.

Solution

$y - y_1 = m(x - x_1)$

$y - 2 = -\dfrac{1}{2}(x - 1)$

For the intercept, rearrange this to the form $y = mx + c$.

$2(y - 2) = -(x - 1)$
$2y - 4 = -x + 1$
$2y = -x + 5$
$y = -\dfrac{1}{2}x + \dfrac{5}{2}$

The intercept is $\dfrac{5}{2}$.

Parallel and perpendicular lines

Consider two lines with equations

l_1 $y = m_1 x + c_1$

l_2 $y = m_2 x + c_2$

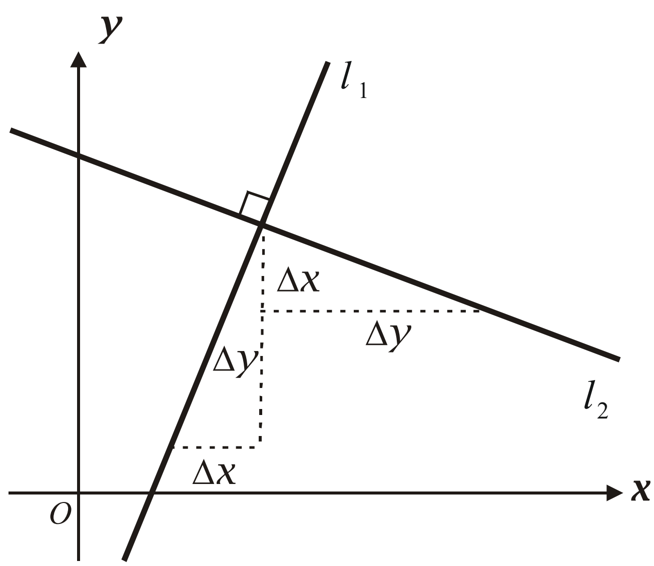
Example (3) showed us that two lines are parallel when their gradients are equal

$m_1 = m_2$.

Furthermore, two lines are perpendicular when their gradients are such that

$m_1 m_2 = -1$.

To prove this, consider the following diagram.



The line l_1 has a positive gradient equal to

$$m_1 = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x}$$

The line l_2 has a negative gradient, which can be found by similar triangles as the diagram shows. From the diagram we see that the ratio of the change in the x-direction to the change in the y-direction is given by

$$m_2 = -\frac{\Delta x}{\Delta y}$$

Thus

$$m_1 \times m_2 = \frac{\Delta y}{\Delta x} \times -\frac{\Delta x}{\Delta y} = -1$$

The *perpendicular bisector* of a line joining two points *A* and *B*, is the line that is perpendicular to *AB* and passes through the midpoint of *AB*.

Example (7)

Find the equation of the perpendicular bisector of the line joining the points $A\left(2, 7/3\right)$ and $B\left(-2/3, 1\right)$

Solution

The gradient of the line passed through A and B is

$$m_1 = \frac{\frac{7}{3} - 1}{2 - \left(-\frac{2}{3}\right)} = \frac{4/3}{8/3} = \frac{1}{2}$$

Let the gradient of the perpendicular bisector be m_2

Then $m_1 m_2 = -1$

So

$m_2 = -2$

Let the equation of the perpendicular bisector be

$y = m_2 x + c_2$

Substituting $m_2 = -2$

$y = -2x + c_2$

mid-point of *AB* is given by

$$\left(x_1, y_1\right) = \left(\frac{\left(2 - \frac{2}{3}\right)}{2}, \frac{\left(\frac{7}{3} + 1\right)}{2}\right) = \left(\frac{2}{3}, \frac{5}{3}\right)$$

Substituting $x = \frac{2}{3},\ y = \frac{5}{3}$ into $y = -2x + c_2$

$\frac{5}{3} = -2 \times \frac{2}{3} + c$

$c = 3$

Hence the equation of the perpendicular bisector is

$y = -2x + 3$

Example (8)

The points *A, B, C* have coordinates $(-3, -1), (6, 2)$ and $(8, k)$ respectively. The line *AB* is perpendicular to *BC*.

(a) Find the gradient of *AB*

(b) Show that $k = -4$

(c) M is the mid-point of BC. Find

(i) the coordinates of M

(ii) the equation of the L, the perpendicular bisector of BC giving your answer in the form $ay + bx + c = 0$

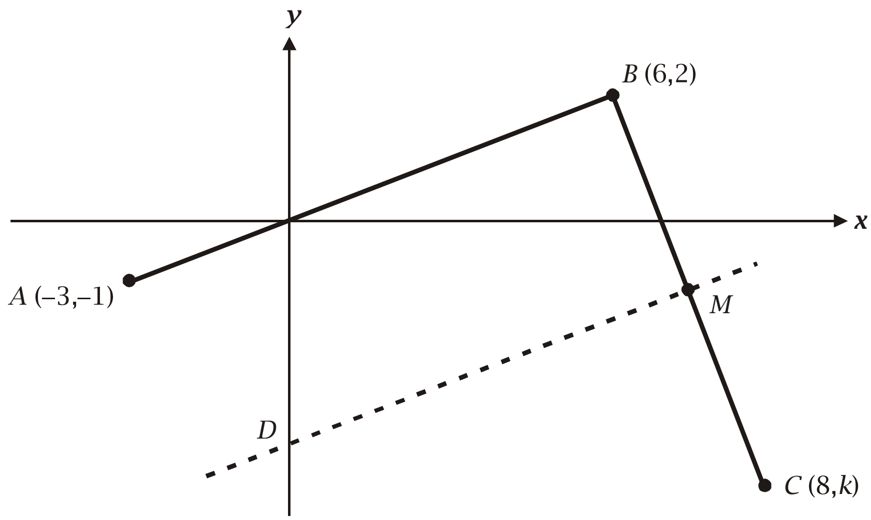
(d) L intersects the y – axis at the point D. Find

(i) the coordinates of D

(ii) the length of MD

Solution

A sketch is a good idea.



(a) $m_1 = \dfrac{\Delta y}{\Delta x} = \dfrac{2-(-1)}{6-(-3)} = \dfrac{3}{9} = \dfrac{1}{3}$

(b) Let the gradient of $BC = m_2$

Since BC is perpendicular to AB

$m_1 m_2 = -1$

$m_2 = -3$

Let the equation of BC be $y = m_2 x + c = -3x + c$

We know that this line passes through $\mathrm{B}(6,2)$. Therefore

$2 = -3 \times 6 + c$

$c = 20$

Hence the equation of BC is

$y = -3x + 20$

At C we have $x = 8$, $y = k$. Hence

$k = -3 \times 8 + 20 = -4$

(c) (i) $M = (\bar{x}, \bar{y}) = \left(\frac{6+8}{2}, \frac{2+(-4)}{2}\right) = (7, -1)$

(ii) L is parallel to AB, so has gradient $m_3 = \frac{1}{3}$ and

equation $y = \frac{1}{3}x + c$. L passes through $M(7, -1)$.

Hence

$-1 = \frac{1}{3} \times 7 + c$

$c = -\frac{10}{3}$

L: $y = \frac{1}{3}x - \frac{10}{3}$ or $3y - x + 10 = 0$

(d) (i) The equation of L is $y = \frac{1}{3}x - \frac{10}{3}$.

At the $y-$axis, $x = 0$, so $y = -\frac{10}{3}$.

$D = \left(0, -\frac{10}{3}\right)$

(ii) The length is

$$MD = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

$$= \sqrt{7^2 + \left(\frac{7}{3}\right)^2}$$

$$= \frac{7}{3}\sqrt{10}$$