The Newton-Raphson Method

The Newton-Raphson formula

Mathematicians seek fast numerical methods of finding solutions (that is, roots) to equations $f(x)=0$. The basis of an iterative technique is that each successive approximation to the root is found by substituting the last approximation into a formula where

$x_{n+1} =$ an iterative process applied to x_n.

You should already be aware that iterative processes do not always converge successfully on a solution. When they do converge different processes may be "faster" than others in the sense that they require fewer iterations to reach an answer to the required degree of accuracy. One very fast iterative process is the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The use of this formula is best illustrated by means of a worked example.

Example (1)

Given $f(x)=x^4+6x+1$ a root to this equation lies between –1 and 0.

(i) Use linear interpolation to find an approximation to this root.

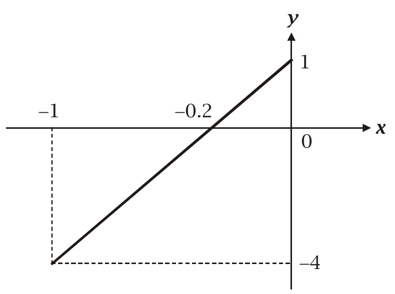
(ii) Use the Newton-Raphson method to find the root giving your answer to 6 d.p.

Solution

(i) We must begin by finding a first approximation to the root to act as a starting value. We do this by linear interpolation, finding the values of $f(-1)$ and $f(0)$ as

$$f(-1)=1-6+1=-4$$
$$f(0)=1$$

We approximate the function $f(x)$ by the straight line joining $(-1,4)$ and $(0,1)$. The phrase "linear interpolation" means, "use a straight line approximation to find a value".



This value is found by ratios in similar triangles and is –0.2. Then

$$f(x) = x^4 + 6x + 1$$

$$f'(x) = 4x^3 + 6$$

and on substituting into the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

we get

$$x_{n+1} = x_n - \left\{ \frac{x_n^4 + 6x_n + 1}{4x_n^3 + 6} \right\}$$

as the iteration formula. Starting with $x_1 = -0.2$ this generates the sequence

$x_1 = -0.2$

$x_2 = -0.16675....$

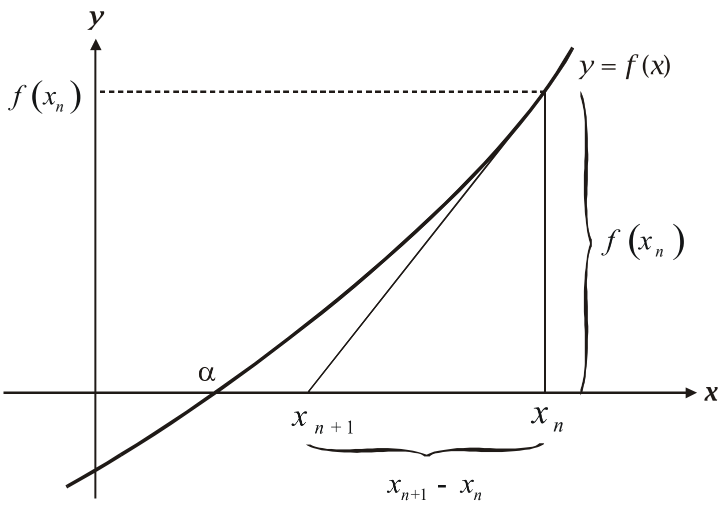
$x_3 = -0.166759566....$

$x_4 = -0.166759566....$

$\therefore \alpha = -0.166796$ (6 d.p.)

Convergence and failure of convergence

However, like most iterative methods, the Newton-Raphson method can fail to converge. To explain why we will first use a graph to explain the meaning of the Newton-Raphson method.



In this diagram, α denotes the root of $y = f(x)$. The $(n+1)$th value, x_{n+1} is found by the intersection of the tangent to the graph of $f(x)$ at the point $(x_n, f(x_n))$. Then the gradient of the tangent is $f'(x_n)$ and is given by

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

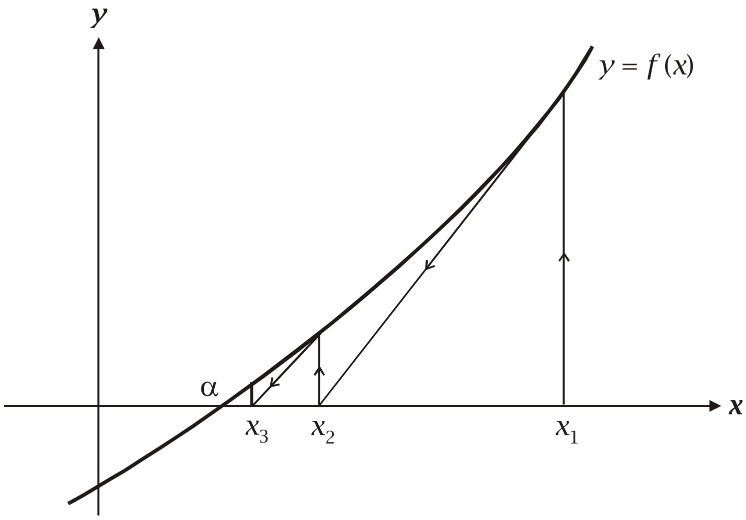
Rearrangement of this gives

$$x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$$

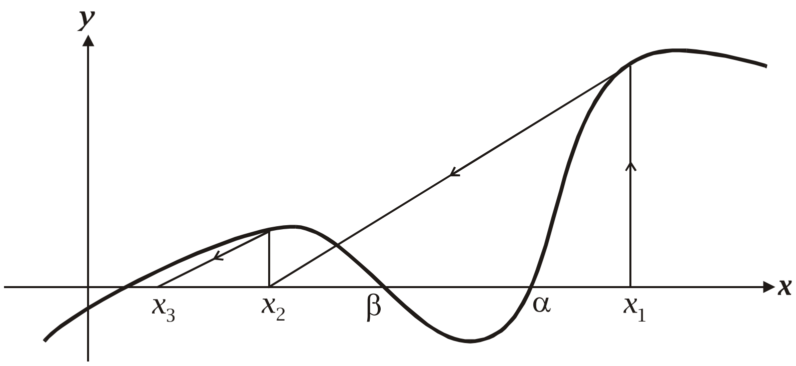
and hence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

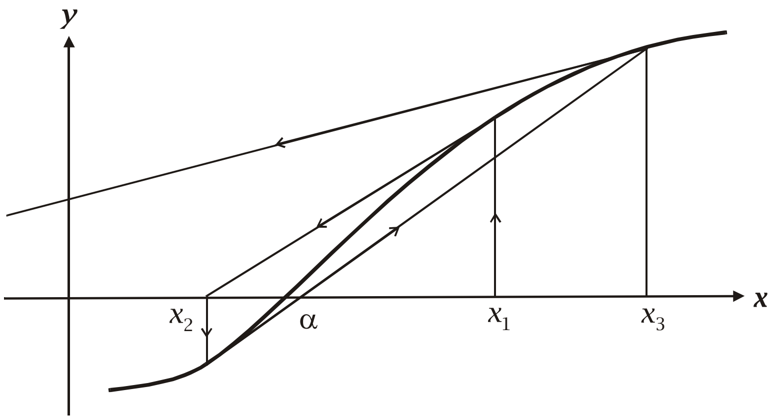
which is the Newton-Raphson formula. The idea is that successive approximations will "home in" on this root – you can see in the above diagram that x_{n+1} is closer to α than x_n. The following graph further illustrates the process of successful convergence



However, convergence depends on the shape of the graph. Hence, there can be failure of convergence. In the following case the method misses out two roots.



In other cases the method is divergent.



Thus, there can be a failure of convergence with this method. You can see from these diagrams that the issue of whether the Newton-Raphson method converges is complex, and it is not usual to discuss at this level precise criteria for convergence.