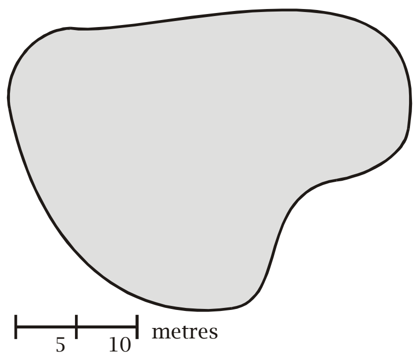
The Trapezium Method

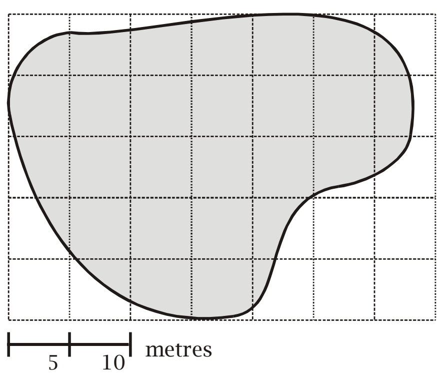
Finding areas

The *trapezium method* is a method for finding areas.

Example (1)
You are asked to find the approximate area of a lawn. You are given the following plan of the lawn. Devise a method for finding the approximate area. Find an approximation and state how your approximation could be improved.

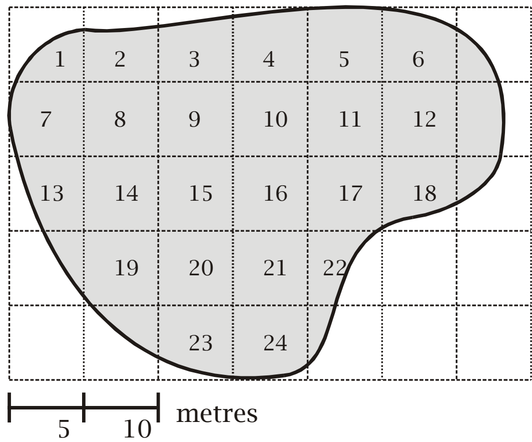


Solution
Divide the shape up into squares – say squares of 5 m length.



The rule is that a square will be counted in if it is more than half covered by lawn; otherwise it will not be counted.

24 squares as "in".



Each square has an area of $5 \times 5 = 25\ m^2$, so the approximate area is

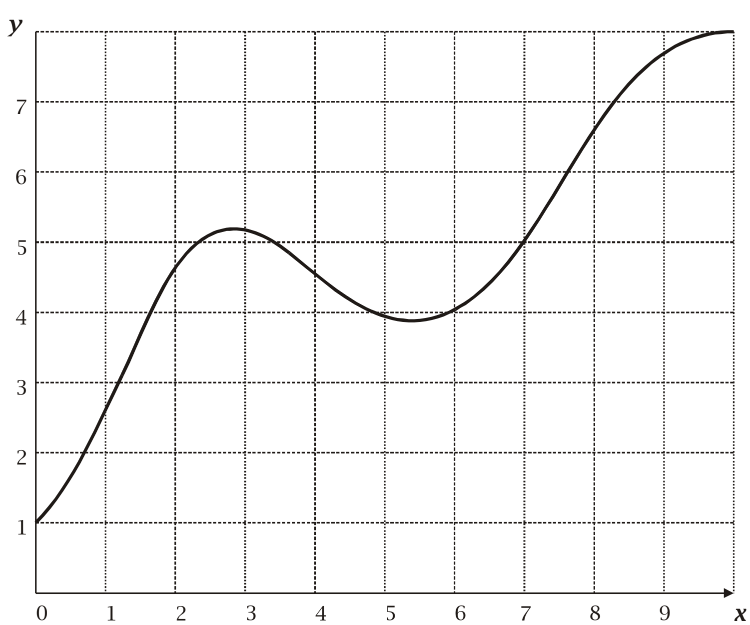
$A \approx 24 \times 25 = 600\ m^2$

The approximation can be improved by taking smaller and smaller squares.

The same method can be applied to areas of curves in graphs.

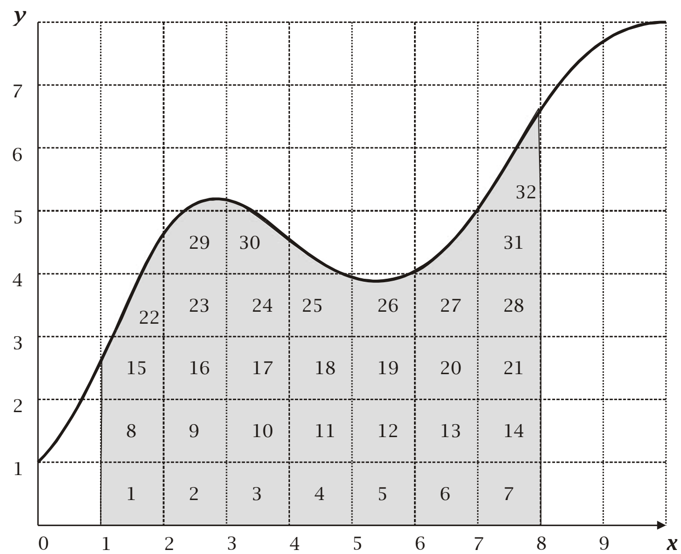
Example (2)

Find the area underneath the curve between $x = 1$ and $x = 8$ in the following diagram.



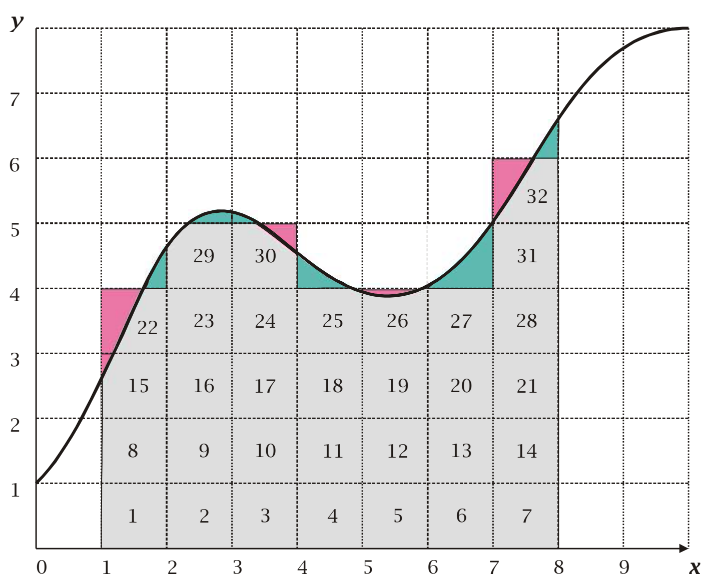
Solution

Using the same method we count the squares underneath the curve between $x = 1$ and $x = 8$. Since each square has an area of 1 square unit, the number of squares gives the approximate area.



So the approximate area is 32 square units.

This approximation is not very accurate. It is difficult to be sure that the areas left out from the approximation are roughly equal to the areas added in. In the following diagram the areas lost and the areas gained are shown in green and red respectively.



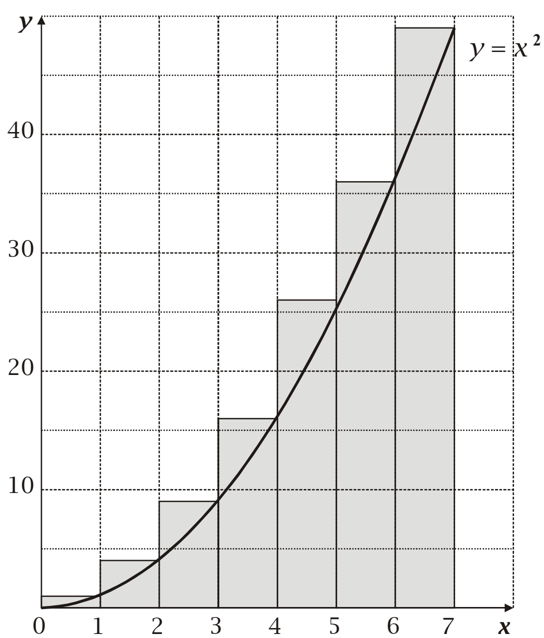
One way to improve the situation is to use smaller squares. However, the disadvantage of this is that the process of counting the squares becomes more tedious. With too many squares to count, uncertainties could creep into the process. Furthermore, in mathematics, we describe graphs by functions. For example, the function

$f(x) = x^2$

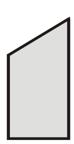
has graph

$y = x^2$

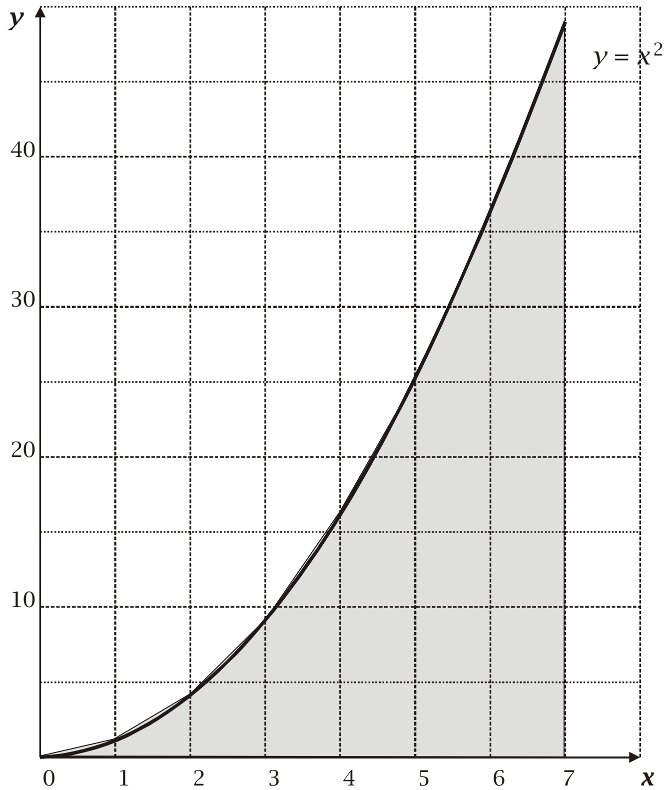
What this means is that for a given value of x we know (or can find) the value of y. In this case squares become unnecessary. We can use rectangles instead. For example, with $y = x^2$ we can use the following rectangles.



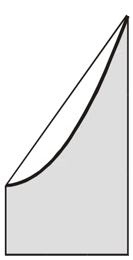
To count the areas of the rectangles is to approximate the area under the curve. To find an area under a curve is called *integration*, and to use an approximate method is a form of *numerical integration*. However, clearly this approximation is (*a*) an overestimate and (*b*) inaccurate, since the areas of the rectangles included lying outside the area bounded by the curve is large relative to the area as a whole. We could improve the accuracy of the approximation by using rectangles of smaller width. Another more efficient way to improve the accuracy of the approximation is to use trapezia instead. A trapezium is a four-sided figure with two parallel sides.



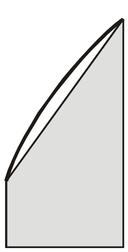
Using trapezia instead of rectangles we obtain the following diagram.



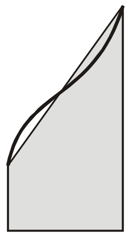
This is still an over-estimate of the area, but the lines are very much closer to the curve. You may not be able to see what is happening in the above diagram, because the lines are so much closer. The following diagram exaggerates how the trapezium is fitted to the curve.



In this case we can see that this is an over-estimate. Sometimes the line fits under the curve.



It is also possible for the curve to change its direction, so that the line lies partly above and partly below the curve.



Whichever way, the move from approximating by rectangles to approximating by trapezia has improved the accuracy of the approximation without much extra cost in calculation. We need to proceed to those calculations. Firstly, before we do so, we introduce some new terminology. Let $y = f(x)$ be a function. Let $x_0, x_1, x_2, \dots x_n$ be a sequence of $n + 1$ successive values of x separated by equal intervals. For example, in the sequence

$x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2 \quad x_3 = 2.5 \quad x_4 = 3$

the difference between each successive value of x is 0.5. Then we call $x_1, x_2, x_3, \dots x_n$ the x *ordinates*. The corresponding y values, $y_0, y_1, y_2, \dots y_n$ where $y = f(x)$ are called the y *ordinates*.

Example (3)

For the function

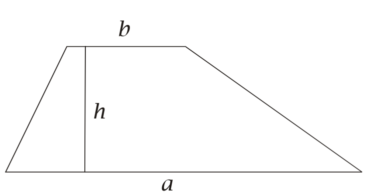
$y = f(x) = x^2 + 1$

find the x and y ordinates for 5 terms separated by intervals of 0.5 starting with $x_0 = 1$.

Solution

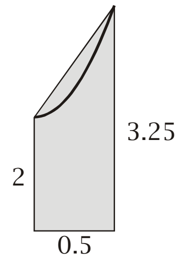
$x_0 = 1$	$y_0 = 2$
$x_1 = 1.5$	$y_1 = 3.25$
$x_2 = 2$	$y_2 = 5$
$x_3 = 2.5$	$y_3 = 7.25$
$x_4 = 3$	$y_4 = 10$

The *area of a trapezium* is $A = \frac{1}{2}(a+b) \times h$ where a and b are the lengths of the two parallel sides and h is the perpendicular distance between them.



Example (4)

Find the area of the following trapezium



Solution

The area is found by taking the average of the two parallel sides and multiplying by the base.

$$A = \frac{1}{2}(2 + 3.25) \times 0.5 = 1.3125 \text{ sq units}$$

We can apply this immediately to the problem of finding the area under a curve.

Example (5)

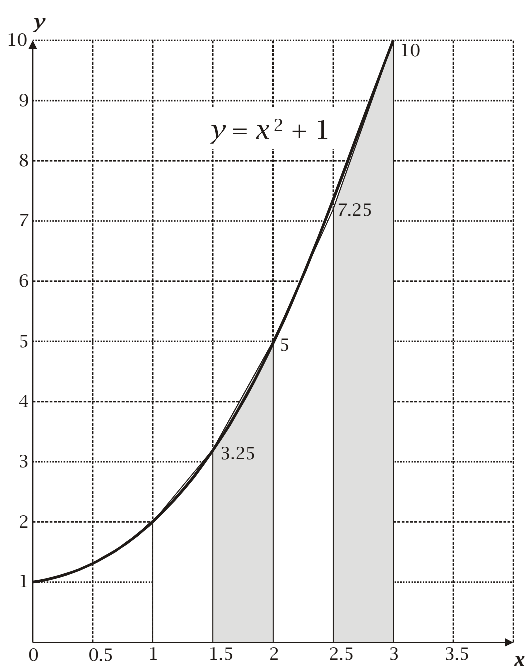
Using trapezia of width 0.5 units find the approximate area under the curve

$y = x^2 + 1$

between $x = 1$ and $x = 2$.

Solution

The following diagram illustrates the solution to the problem.

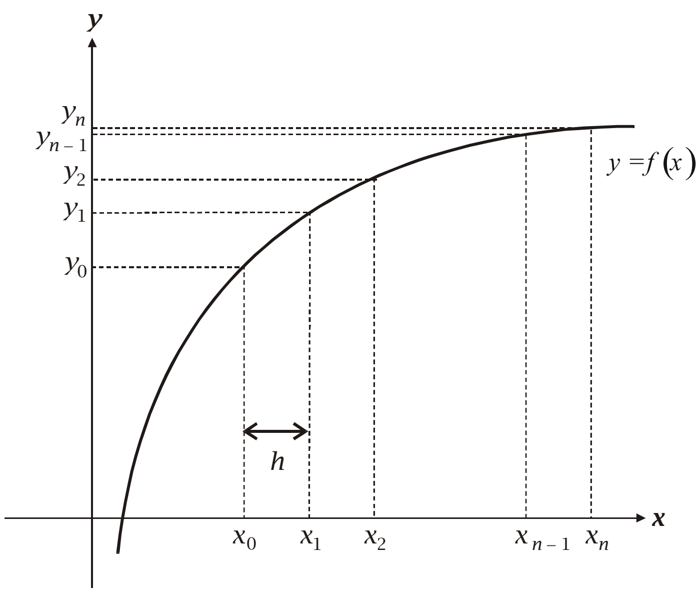


So the approximation is

$$\begin{aligned}\text{Total area} &\approx \text{area of 1st } + \text{area of 2nd } + \text{area of 3rd } + \text{area of 4th trapezium}\\ &= 0.5\times\frac{1}{2}(2+3.25)+0.5\times\frac{1}{2}(3.25+5)+0.5\times\frac{1}{2}(5+7.25)+0.5\times\frac{1}{2}(7.25+10)\\ &= 1.3125+2.0625+3.0625+4.3125\\ &= 10.75 \text{ sq units}\end{aligned}$$

The trapezium rule

The trapezium rule is a general formula for approximating the area under the graph of the function $y = f(x)$ by trapezia of width h between ordinates x_0 and x_n. The following diagram illustrates this idea.



The width of each trapezium is h.

The area of each trapezium is $\frac{h}{2}(y_k + y_{k+1})$.

The sum the areas of the trapezia is

$$A \approx \frac{h}{2}(y_0 + y_n) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

In this formula the middle ordinates (the $y_1 \ldots y_{n-1}$ values) appear twice and the end ordinates (y_1 and y_n) appear once. So the trapezium rule is

$$A = \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

In words the rule is: sum the first and last ordinates, add twice the sum of the remaining ordinates, and multiply the whole lot by $\frac{1}{2}$ the width. The width is given by $h = \frac{b-a}{n}$ where a and b are the values of the first and last x ordinates respectively and there are n intervals (giving $n+1$ ordinates).

Example (6)

The equation of a curve is $y = x^2 - 3x + 7$.

(*a*) By completing the square, sketch the curve.

(*b*) A region R is bounded by the curve, the axes and the ordinate at $x = 5$. Divide the region R into five trapezia of equal widths and add these to the graph sketched in (*a*).

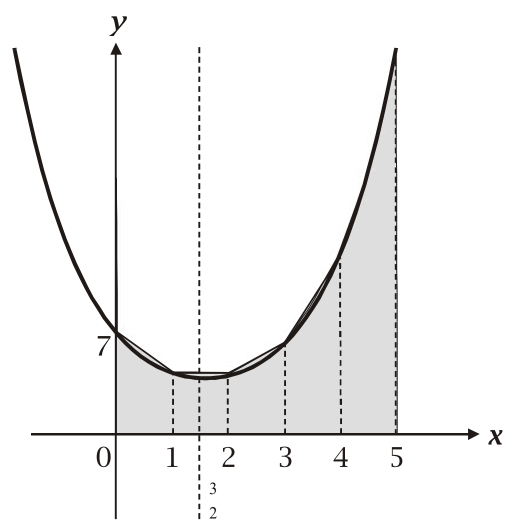
(*c*) Use the trapezium rule, with five strips of equal width, to find an approximate area of R.

Solution

(*a*) (*b*) The completed square form gives

$$\begin{aligned} y &= x^2 - 3x + 7 \\ &= x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 7 \\ &= \left(x - \frac{3}{2}\right)^2 + \frac{19}{4} \end{aligned}$$

So the curve has axis of symmetry at $x = \frac{3}{2}$ and minimum value $y = \frac{19}{4}$. The intercept with the y-axis is at $y = 7$.



(*c*) This area will be approximated by the trapezium method, with formula

$$A \approx \frac{h}{2}\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$$

The ordinates are

$y_0 = y(0) = 7$

$y_1 = y(1) = 1 - 3 + 7 = 5$

$y_2 = y(2) = 4 - 6 + 7 = 5$

$y_3 = y(3) = 9 - 9 + 7 = 7$

$y_4 = y(4) = 16 - 12 + 7 = 11$

$y_5 = y(5) = 25 - 15 + 7 = 17$

Hence

$$A \approx \frac{1}{2}\{7 + 17 + 2(5 + 5 + 7 + 11)\} = 40 \text{ sq. units}$$

What we have been finding is an approximation to the area under a curve $y = f(x)$ between a starting ordinate $a = x_0$ and a finishing ordinate $b = x_n$. The *exact* area under the curve $y = f(x)$ from *a* to *b* is denoted by the symbol

$$I = \int_a^b f(x)\,dx$$

It is called the *exact integral* of $y = f(x)$ from *a* to *b*.

Example (7)

Use the trapezium rule with six ordinates to find an approximate value for

$$\int_0^1 \sqrt{\frac{1}{2 + x^3}}\,dx$$

Show your working and give your answer to four significant figures.

Solution

Here $y = f(x) = \sqrt{\frac{1}{2 + x^3}}$ and we are being asked to approximate the area under this curve between the lines $x = 0$ and $x = 1$. There are five trapezia involved in the approximation and six ordinates at $x_0 = 0$ $x_1 = 0.2$ $x_2 = 0.4$ $x_3 = 0.6$ $x_4 = 0.6$ $x_5 = 1.0$. The interval width is $h = 0.2$. The corresponding *y*-ordinates to 6 significant figures are

$y_0 = 0.707107$

$y_1 = 0.705697$

$y_2 = 0.696058$

$y_3 = 0.671762$

$y_4 = 0.630943$

$y_5 = 0.577350$

Substituting into the trapezium rule

$$
\begin{aligned}
A &\approx \frac{h}{2}\left\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right\} \\
&= \frac{0.2}{2}\left\{0.707107 + 0.577350 + 2(0.705697 + 0.696058 + 0.671762 + 0.630943)\right\} \\
&= 0.6694331 \\
&= 0.6694 \ (4.\text{s.f.})
\end{aligned}
$$