The Uniform Distribution

Prerequisites

You should be familiar with the properties of continuous probability distributions.

(1) **Definition of a probability density function**

Let $f(x)$ be a continuous function. If

(1) $f(x) \geq 0$ for all x

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

then *f (x)* defines a continuous *probability density function*. The condition $f(x) \geq 0$ for all x is necessary because we cannot have negative probabilities. Note that the expression $\int_{-\infty}^{\infty} f(x)\, dx = 1$ may also be written $\int_{\text{all } x} f(x) dx = 1$. It says that the total area under the probability density function $f(x)$ for all values of x is equal to 1.

(2) **Expectation and variance of a continuous probability distribution**

If X is a continuous random variable having probability density function $f(x)$ then the expectation is $E(x) = \int_{-\infty}^{\infty} x f(x) dx$. The expectation $E(x)$ may also be called the **mean** of the distribution. We also have

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$ $\qquad \text{var}(X) = E(X^2) - [E(X)]^2$

where $\text{var}(X)$ is the variance of the distribution.

(3) **Cumulative distribution function**

If X is a random variable with probability density function $f(x)$ then the cumulative distribution function of X is given by $F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x) dx$.

(4) **Median**

Let m denote the median value. Then, if $F(t)$ is the cumulative distribution function for X, $F(m) = \frac{1}{2}$. If $f(x)$ is the probability density function for X then $\int_{-\infty}^{m} f(x) dx = \frac{1}{2}$.

(5) **Quartiles, percentiles**

The first (lower) and third (upper) quartiles are the values t_1 and t_3 such that

$$F(t_1) = \frac{1}{4} \text{ and } F(t_3) = \frac{3}{4}$$

where $F(t)$ is the cumulative probability function. A percentile is a value of t such that $F(t)$ is equal to a given percentage of the cumulative distribution.

Example (1)

The continuous random variable *X* has probability density function *f* given by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{8}{3}x + \frac{1}{3} & \text{for } 0 \le x \le t \\ 0 & \text{for } x > t \end{cases}$$

(*a*) Find *t*.

(*b*) Find the cumulative distribution function for *X*.

(*c*) Evaluate $P\left(\frac{1}{4} \le X \le \frac{1}{3}\right)$.

(*d*) Find the median of *X*.

(*e*) Find the mean and variance of *X*.

Solution

(*a*) By the law of total probability

$$\int_0^t f(x)\,dx = 1$$

Hence

$$\int_0^t \frac{8}{3}x + \frac{1}{3}\,dx = 1$$

$$\left[\frac{4}{3}x^2 + \frac{1}{3}x\right]_0^t = 1$$

$$\frac{1}{3}\left(4t^2 + t\right) = 1$$

$$4t^2 + t - 3 = 0$$

$$(4t - 3)(t + 1) = 0$$

$$t = -1 \text{ or } t = \frac{3}{4}$$

t cannot be negative; therefore $t = \frac{3}{4}$

(*b*) The cumulative distribution function on the interval for $0 \le x \le \frac{3}{4}$ is

$$F(t) = \int_0^t f(x)\,dx = \frac{4}{3}t^2 + \frac{1}{3}t$$

We write the cumulative distribution function in terms of the original variable and define it piecewise as follows.

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{3}\left(4x^2 + x\right) & \text{for } 0 \le x \le \frac{3}{4} \\ 1 & \text{otherwise} \end{cases}$$

(*c*) $F\left(\frac{1}{4}\right) = \frac{1}{3}\left(4\left(\frac{1}{4}\right)^2 + \frac{1}{4}\right) = \frac{1}{6}$ $\qquad F\left(\frac{1}{3}\right) = \frac{1}{3}\left(4\left(\frac{1}{3}\right)^2 + \frac{1}{3}\right) = \frac{7}{27}$

$$P\left(\frac{1}{4} \le X \le \frac{1}{3}\right) = F\left(\frac{1}{3}\right) - F\left(\frac{1}{4}\right) = \frac{7}{27} - \frac{1}{6} = \frac{5}{54}$$

(*d*) Let m denote the median, then $F(m) = \frac{1}{2}$

$\frac{1}{3}\left(4m^2 + m\right) = \frac{1}{2}$ $\qquad 8m^2 + 2m - 3 = 0$ $\qquad (4m+3)(2m-1) = 0$

$m = \frac{1}{2}$ since m cannot be negative

(*e*) $E(x) = \int_0^{\frac{3}{4}} x f(x)\,dx = \int_0^{\frac{3}{4}} x\left(\frac{8}{3}x + \frac{1}{3}\right)dx = \int_0^{\frac{3}{4}} \frac{8}{3}x^2 + \frac{1}{3}x\,dx = \left[\frac{8}{9}x^3 + \frac{1}{6}x^2\right]_0^{\frac{3}{4}} = \frac{15}{32}$

$$E\left(X^2\right) = \int_0^{\frac{3}{4}} x^2 f(x)\,dx = \int_0^{\frac{3}{4}} \frac{8}{3}x^3 + \frac{1}{3}x^2\,dx = \left[\frac{2}{3}x^4 + \frac{1}{9}x^3\right]_0^{\frac{3}{4}} = \frac{33}{128}$$

$$Var(X) = E\left(X^2\right) - \left[E(X)\right]^2 = \frac{33}{128} - \left(\frac{15}{32}\right)^2 = \frac{39}{1024}$$

The uniform distribution

Definition of the uniform distribution

Let X be a continuous random variable with probability density function

$f(x) = \frac{1}{b-a}$ for $a \le x \le b$

$f(x) = 0$ otherwise

where a and b are constants. Then X is said to follow a *uniform distribution.* This is denoted by $X \sim R(a,b)$. The uniform distribution may also be referred to as the *rectangular distribution.*

Example (2)

Prove that the function

$f(x) = \frac{1}{b-a}$ for $a \le x \le b$ a, b constants

$f(x) = 0$ otherwise

defines a probability density function.

Solution

We have to show

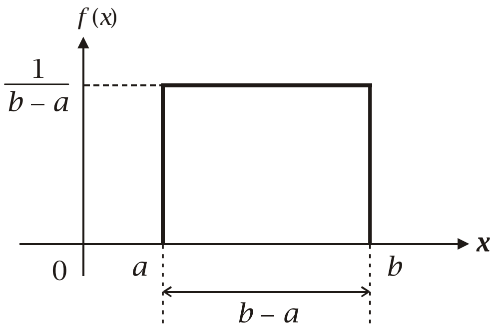
(1) $f(x) \ge 0$ for all x

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

For (1), since $a \le x \le b$, then $b - a > 0$ and $f(x) = \frac{1}{b-a} > 0$ for all x

For (2) $\int_{-\infty}^{\infty} f(x)\, dx = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a}[x]_a^b = \frac{b-a}{b-a} = 1$

The uniform distribution derives its name from the fact that the probability density function takes the same value for all values x lying in the interval $a \le x \le b$.



Expectation and variance of a uniform distribution

The mean (or expectation) and variance of a uniform distribution, $X \sim R(a,b)$, are given by

$E(X) = \frac{1}{2}(a+b)$

$\text{var}(X) = \frac{1}{12}(b-a)^2$

Example (3)

The random variable X is uniformly distributed on the interval $[a,b]$. The mean of X is 0 and the variance of X is $\frac{\pi^2}{3}$.

(a) Find a and b.

(b) Find the probability density function of X.

(c) Sketch the distribution of X.

Solution

(a) $E(X)=\frac{1}{2}(a+b) \quad \Rightarrow \quad \frac{1}{2}(a+b)=0 \quad \Rightarrow \quad a=-b$

$\text{var}(X)=\frac{1}{12}(b-a)^2$

$\frac{1}{12}(b-a)^2=\frac{\pi^2}{3}$

$(b-a)^2=4\pi^2$

$b-a=2\pi$ $\quad$ $[b-a$ cannot be negative$]$

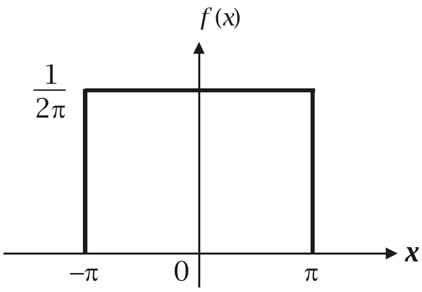
Since $a=-b$

$b=\pi \quad a=-\pi$

(b) $f(x)=\frac{1}{2\pi}$ $\quad$ for $-\pi \le x \le \pi$

$f(x)=0$ $\quad$ otherwise

(c)



Example (4)

Let $X \sim R(a,b)$. Prove $E(X)=\frac{1}{2}(a+b)$

Solution

$$E(x)=\int_a^b x\left(\frac{1}{b-a}\right)dx=\frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b=\frac{1}{2(b-a)}\left(b^2-a^2\right)=\frac{1}{2(b-a)}(b-a)(b+a)=\frac{1}{2}(b+a)$$

Example (5)

(a) Show that $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$

(b) Let $X \sim R(a,b)$. Prove $\operatorname{var}(X) = \frac{1}{12}(b-a)^2$

Solution

(a) $(b-a)(b^2+ab+a^2) = b^3 + ab^2 + ba^2 - ab^2 - a^2b - a^3 = b^3 - a^3$

(b)

$$E(X^2) = \int_a^b x^2\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b$$

$$= \frac{1}{3(b-a)}(b^3 - a^3)$$

$$= \frac{1}{3(b-a)}(b-a)(b^2+ab+a^2) \qquad \text{[By part } (a)\text{]}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\operatorname{var}(X) = E(X^2) - [E(X)]^2$$

$$= \left(\frac{b^2+ab+a^2}{3}\right) - \left(\frac{1}{2}(b+a)\right)^2$$

$$= \left(\frac{b^2+ab+a^2}{3}\right) - \left(\frac{b^2+2ab+a^2}{4}\right)$$

$$= \frac{1}{12}\{4b^2+4ab+4a^2-3b^2-6ab-3a^2\} = \frac{1}{12}(b^2-2ab+a^2) = \frac{1}{12}(b-a)^2$$

Expectation and variance of a function of a uniformly distributed variable

Let us suppose that the drawing of a geometric shape depends on the random selection of some variable, such as the side of one of its lengths or an angle. Let this variable be uniformly distributed. As a result further questions about the probability of the shape having a certain area may arise. For example a square has a side X cm. Its area is a function of the length of the square given by $g(X) = X^2$. Suppose that X is a continuous random variable that is uniformly distributed on the interval $[a,b]$. This says that the length of the side of the square is chosen at random as a number lying in the interval $[a,b]$, and that every number is equally likely. We would like to know the expected mean of the area given by $g(X)=X^2$. Actually, we already know this. It is $E(X^2)=\int_a^b x^2\left(\frac{1}{b-a}\right)dx$.

Example (6)

A square has side *L* cm, where *L* is a continuous random variable that is uniformly distributed on the interval $[2,8]$.

(*a*) State the probability density function of *L*.

(*b*) Find the expected area of the square.

(*c*) Find the probability that the area of the square is less than 45 cm², giving your answer correct to three decimal places.

Solution

(*a*) $L \sim R(2,8)$

The probability density function is given by

$f(x)=\frac{1}{8-2}=\frac{1}{6}$ for $2 \le x \le 8$

$f(x)=0$ otherwise

(*b*) Using the result:

$$E(X^2)=\int_a^b x^2\left(\frac{1}{b-a}\right)dx$$

Here $a=2$, $b=8$

$$E(X^2)=\int_2^8 x^2\left(\frac{1}{6}\right)dx=\frac{1}{6}\left[\frac{1}{3}x^3\right]_2^8=\frac{1}{18}(512-8)=28 \text{ cm}^2$$

(*c*) $P(\text{Area}<45)=P\left(L<3\sqrt{5}\right)=\int_2^{3\sqrt{5}}\frac{1}{6}dx=\frac{1}{6}[x]_2^{3\sqrt{5}}=\frac{1}{6}\left(3\sqrt{5}-2\right)=0.785 \text{ (3 d.p.)}$

In the above example we had $g(x)=x^2$ and $E(X^2)=\int_a^b g(x)\left(\frac{1}{b-a}\right)dx=\int_a^b x^2\left(\frac{1}{b-a}\right)dx$. We can generalise this to any function *g* of the random variable *X*. In the formula $E(X)=\int_{\text{all } x} x f(x)dx$ we are finding the integral of the product of the value of *x* and its probability density function $f(x)$. We can write this as

$E(X) = \text{Integrate for all values}[\text{value} \times \text{probability density}]$

When the value is given by another function $g(x)$ then this becomes

$E(g(X)) = \text{Integrate for all } x[g(x) \times \text{probability density}]$.

The probability density is a function $f(x)$, so we have in general $E(g(X)) = \int_{\text{all } x} g(x)f(x)\,dx$.

Expectation of any function of a uniformly distributed random variable *X*

In the case of the uniform distribution the probability density function is

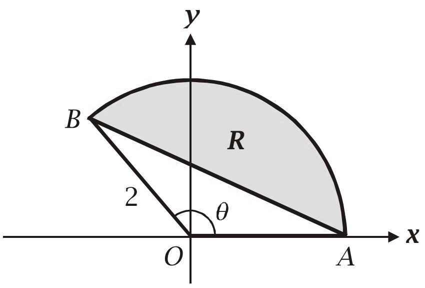
$$f(x) = \frac{1}{b-a} \qquad \text{for } a \le x \le b$$

Hence

$$E(g(X)) = \int_a^b g(x)\left(\frac{1}{b-a}\right)dx$$

Example (7)

In the diagram the shaded area R is a segment of a sector of a circle subtending an angle θ radians at the centre. The radius of the circle is 2 cm. The angle θ is a uniformly distributed random variable defined on the interval $[0,\pi]$



(*a*) Write down in full the probability density function of θ.

(*b*) (*i*) Show that the area of R is given by

$R = 2\theta - 2\sin\theta$

(*ii*) Find R when $\theta = \frac{\pi}{4}$

(*iii*) Find $P\left(R \le \frac{\pi}{2} - \sqrt{2}\right)$.

(*c*) Find $E(R)$ giving your answer to three decimal places.

Solution

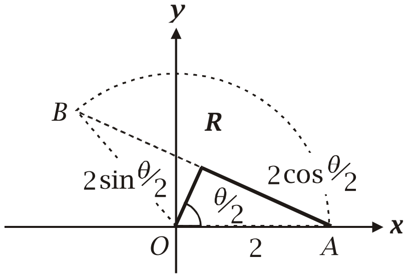
(*a*) $\theta \sim R(0,\pi)$

This has probability density

$f(\theta) = \frac{1}{\pi}$ for $0 \le \theta \le \pi$

$f(\theta) = 0$ otherwise

(*b*) (*i*)



$$R = \text{area}(\text{sector}) - \text{area}\,\Delta(OAB)$$

$$= \frac{1}{2}(2)^2\theta - 4\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$= 2\theta - 2\sin\theta \qquad \left[\text{since } \sin\theta \equiv 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\right]$$

(*ii*) When $\theta = \frac{\pi}{4}$, $R = 2 \times \frac{\pi}{4} - 2\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - \sqrt{2}$.

(*iii*) $P\left(R \le \frac{\pi}{2} - \sqrt{2}\right) = P\left(\theta \le \frac{\pi}{4}\right)$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\pi}\,d\theta$$

$$= \frac{1}{\pi}\left[\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}$$

(*c*) Using the result $E(g(X)) = \int_a^b g(x)\left(\frac{1}{b-a}\right)dx$ with

$g(x) = 2\theta - 2\sin\theta$, $a = 0$ and $b = \pi$

$$E(R) = \frac{1}{\pi}\int_0^{\pi} 2\theta - 2\sin\theta\;dx$$

$$= \frac{1}{\pi}\left[\theta^2 + 2\cos\theta\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(\pi^2 - 2\right) - (2)\right\} = \pi - \frac{4}{\pi} = 1.868 \text{ (3 d.p.)}$$