Transformations of graphs

The completed square form and transformations of the parabola

You should be able to find the completed square form of a quadratic function.

Example (1)

(*a*) Find the completed square form of the quadratic polynomial

$y = 2x^2 - 4x + 5$

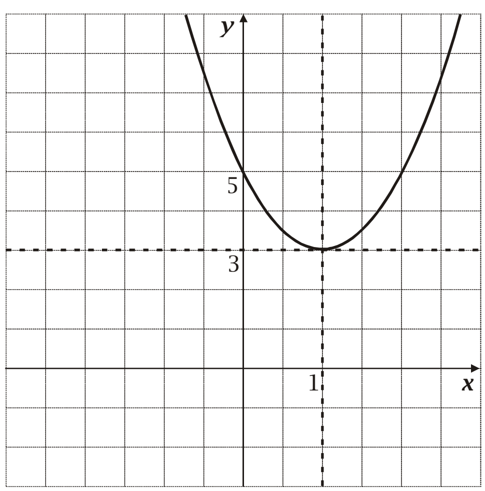
(*b*) Sketch its graph.

Solution

(*a*)

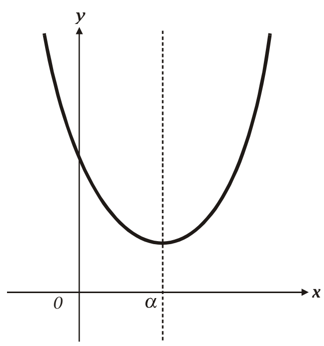
$$\begin{aligned} y &= 2x^2 - 4x + 5 \\ &= 2\left(x^2 - 2x\right) + 5 \\ &= 2\left(x^2 - 2x + (1)^2 - (1)^2\right) + 5 \\ &= 2\left(x^2 - 2x + (1)^2\right) - 2 \times (1)^2 + 5 \\ &= 2(x-1)^2 - 2 + 5 \\ &= 2(x-1)^2 + 3 \end{aligned}$$

(*b*) This has axis of symmetry $x = 1$; the parabola $y = x^2$ has been stretched vertically by a scale factor of +2; its minimum point is at $y = 3$, and from the original equation, $y = 2x^2 - 4x + 5$, we can determine that its intercept on the *y*-axis is $y = 5$. With all this information we can sketch the graph.

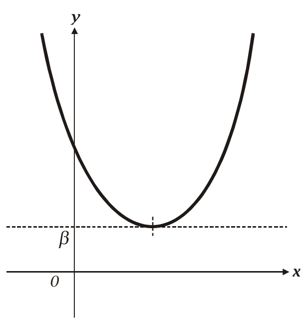


We have already indicated that every quadratic function has the same shape as the basic function $y = x^2$; that is, every quadratic function is a parabola. This means that the graph of any quadratic function can be obtained from the graph of $y = x^2$ by means of *transformations* of that graph. In fact, the completed square form tells us already what those transformations are. In the completed square form $y = \mu(x-\alpha)^2 + \beta$ we see three parameters, α, β and μ.

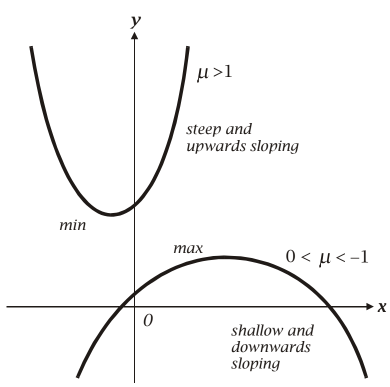
(1) The parameter α gives the vertical *axis of symmetry* of the parabola $y = \mu(x-\alpha)^2 + \beta$.



(2) The parameter β gives the minimum/maximum value of the parabola $y = \mu(x-\alpha)^2 + \beta$.



(3) The parameter μ describes the *scaling* of the parabola $y = \mu(x-\alpha)^2 + \beta$. If $\mu > 1$ then the graph is steeper than $y = x^2$. If $\mu < 1$ then the graph is shallower than $y = x^2$. If $\mu > 0$ (positive) then the graph is orientated upwards and has a minimum value, and if $\mu < 0$ (negative) then the graph is directed downwards and has a maximum value.

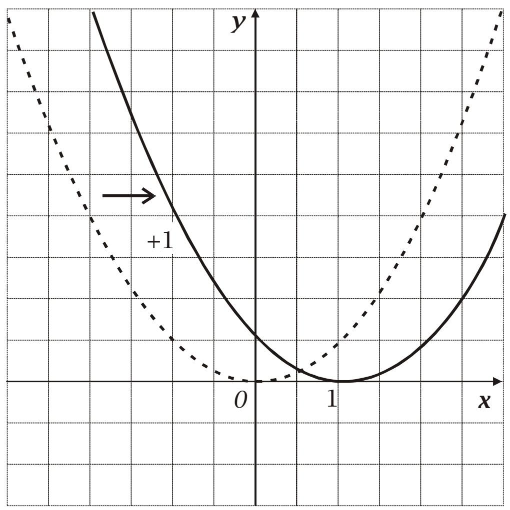


Example (1) continued

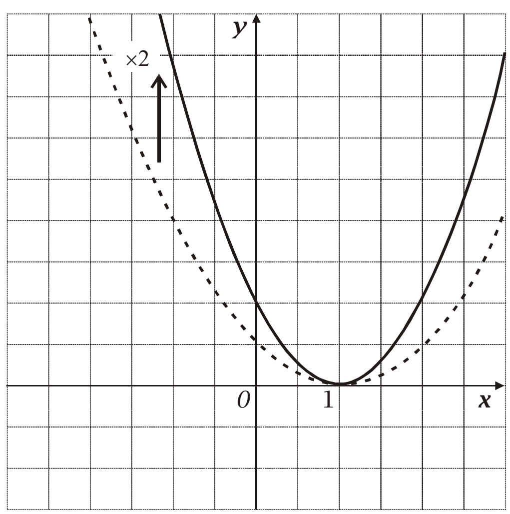
The function $y = 2x^2 - 4x + 5$ can be obtained from the function $y = x^2$ by means of three transformations. What are these?

Solution

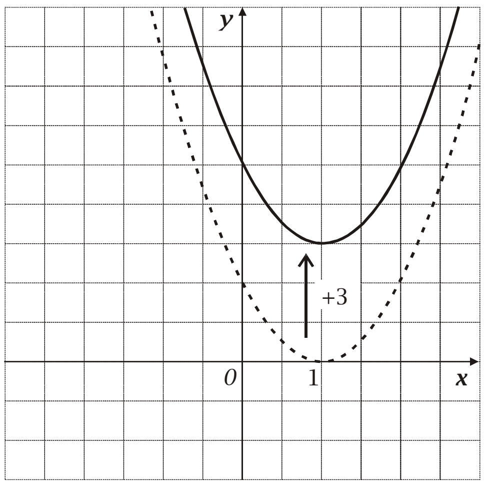
Firstly, a horizontal translation by +1. This takes $y = x^2$ to $y = (x-1)^2$



The second transformation is a scaling by a factor of 2. This stretches the parabola upwards. This takes $y = (x-1)^2$ to $y = 2(x-1)^2$



The third transformation is a vertical translation by +3. It pushes the graph upwards and takes $y = 2(x-1)^2$ to $y = 2(x-1)^2 + 3$ which is the completed square form of the function $y = 2x^2 - 4x + 5$.



The same rules for horizontal and vertical translations and vertical scalings apply to any other curve.

Transformations of graphs

Imagine a shape drawn on a piece of elastic material on top of a table. Without distorting the image we can move and manipulate that picture in several ways.

1. We can move it up or down the table
2. We can move it sideways
3. We can stretch it vertically
4. We can stretch it horizontally
5. We can rotate it
6. We can flip it over.

We call all such manipulations of objects transformations. In geometry objects are described by equations. Geometric objects are graphs corresponding to these equations. The six transformations above are given the following names.

1. Vertical translation
2. Horizontal translation
3. Vertical scaling
4. Horizontal scaling
5. Rotation
6. Reflection

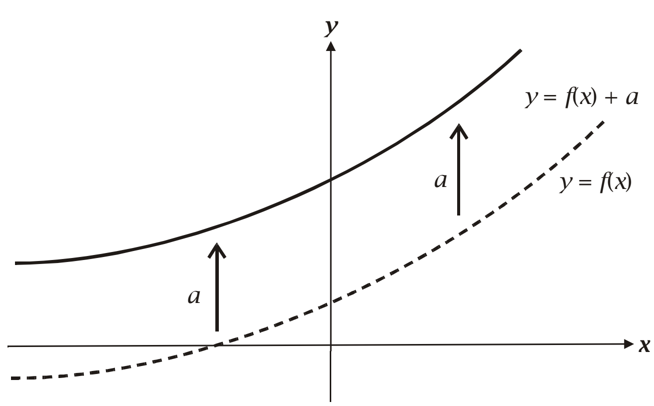
In this chapter we deal only with the first four transformations, and apply them to graphs of functions. A graph is described by a function $y = f(x)$. Therefore, in describing transformations of graphs, we need to think in terms of "doing something" to the expression $y = f(x)$. Before we look in detail at this, a brief note about the difference between y and $f(x)$ in the expression $y = f(x)$; y stands for the *image* of x as a result of applying the function f to x. A graph is a pictorial representation of the relationship between x and y. The function f represents the *process* (or *mapping*) whereby y is obtained from x. The expression $y = f(x)$ says that the image of x under f is obtained by "doing f to x". To all intensive purposes y and $f(x)$ are interchageable. In fact, they are so interchangeable that you will often see the expression $y = y(x)$. On the left of this y is used to represent an image of x, but on the right y is used to represent a function of x.

Vertical translation

The vertical translation of $y = f(x)$ is found by adding a real number a to every point is

$y = f(x) + a$

The effect of a vertical translation on the image y of the function $y = f(x)$ is represented by the mapping diagram $y \to y + a$. In other words, we add a to every value



Example (2)

A cubic function is defined by

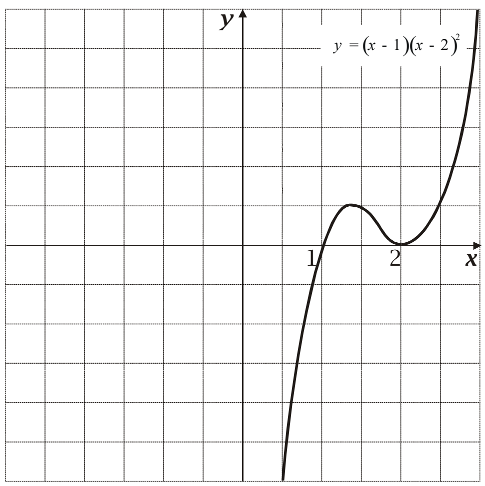
$y = (x-1)(x-2)^2$

(a) Sketch the graph of this function. How many roots does this function have?

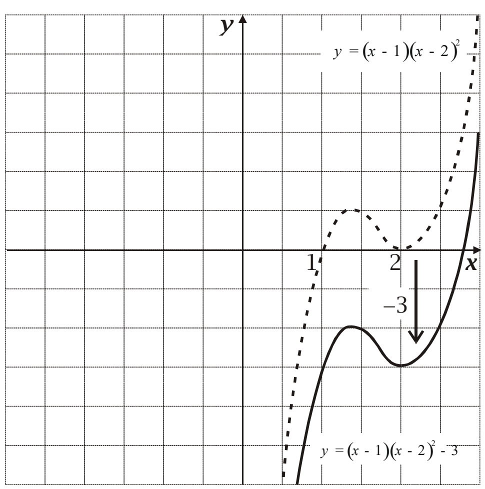
(b) This function undergoes a vertical translation by $y \to y-3$. Sketch the resultant graph. Find the new function that results from this transformation and state how many roots the new function has.

Solution

(*a*) To sketch the graph note that the roots are given by the factors of $y = (x-1)(x-2)^2$. The roots are $x = 1$ and $x = 2$ (twice). The meaning of the double root at $x = 2$ is that the graph just touches the *x*-axis at this point. The function is negative when $x < 1$.



(*b*) We can now show the effect of the vertical translation, downwards, by 3



To find its equation we must first expand and collect the terms of the original function

$$\begin{aligned} y &= (x-1)(x-2)^2 \\ &= (x-1)(x^2-4x+4) \\ &= x^3-4x^2+4x-x^2+4x-4 \\ &= x^3-5x^2+8x-4 \end{aligned}$$

The vertical translation is simply by subtracting 3 from this expression

$y \to y-3$

Giving the new function

$y = x^3-5x^2+8x-7$

The graph indicates that this function will have only one root.

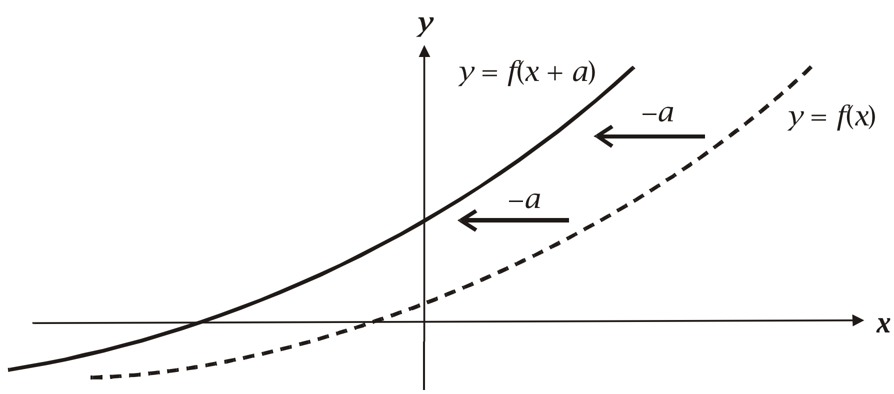
Horizontal Translation

The mapping

$x \to x+a$

shifts the graph of $y = f(x)$ by $-a$ along the *x*-axis. The resultant function is

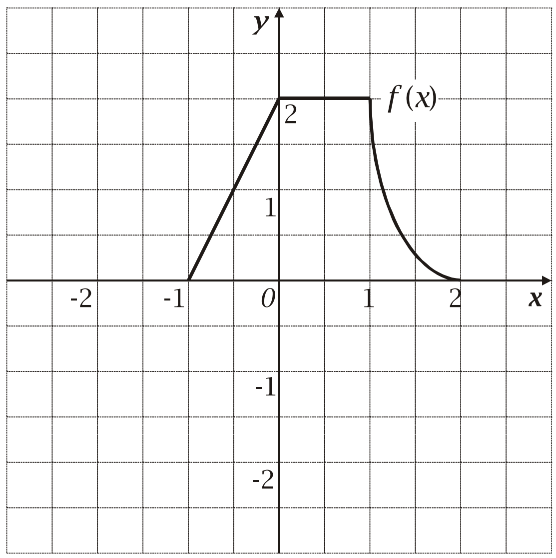
$y = f(x+a)$.



One way to remember the direction in which $y = f(x+a)$ shifts the graph of $f(x)$ is as follows: examining $y = f(x+a)$ consider what value of *x* will make $f(x+a) = f(0)$. This will be $x = -a$. So at $x = -a$ the function $f(x+a)$ will take the same value as $f(0)$. This means that $f(x+a)$ is obtained by shifting the graph of $f(x)$ by $-a$ to the left, in the negative direction.

Example (3)

A function $y = f(x)$ has graph



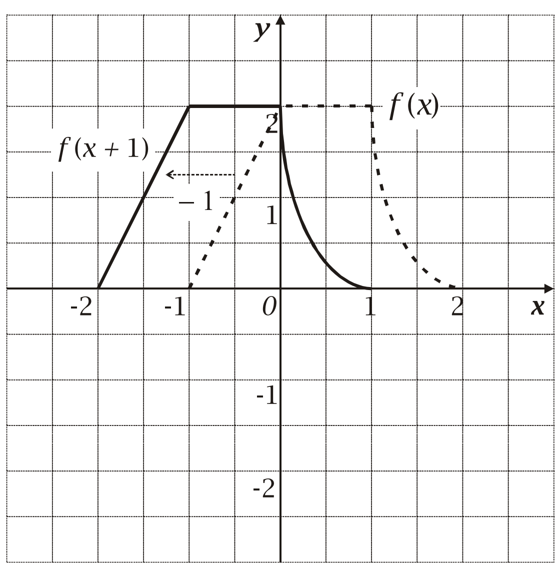
What is the relationship between this function and the function whose equation is $f(x+1)$? Sketch both functions on the same graph.

Solution

$f(x+1)$ is obtained from $y = f(x)$ by a horizontal translation

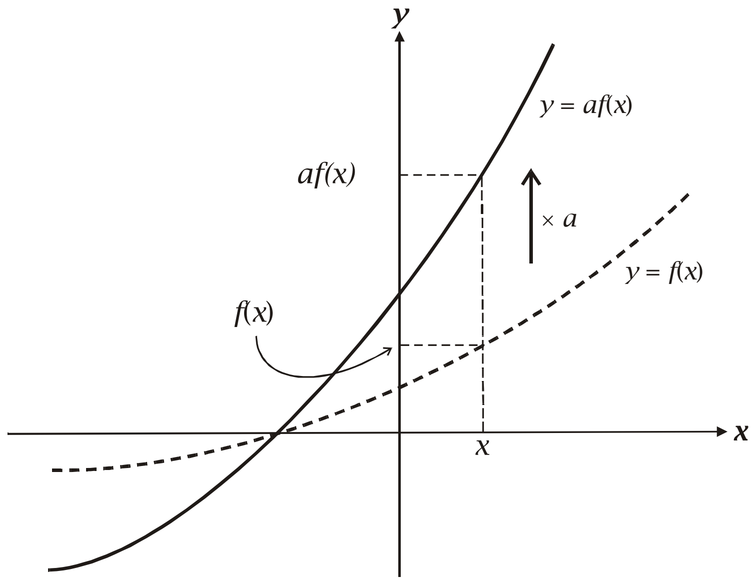
$x \to x - 1$

That is by shifting $y = f(x)$ by –1 to the left (in the negative direction).

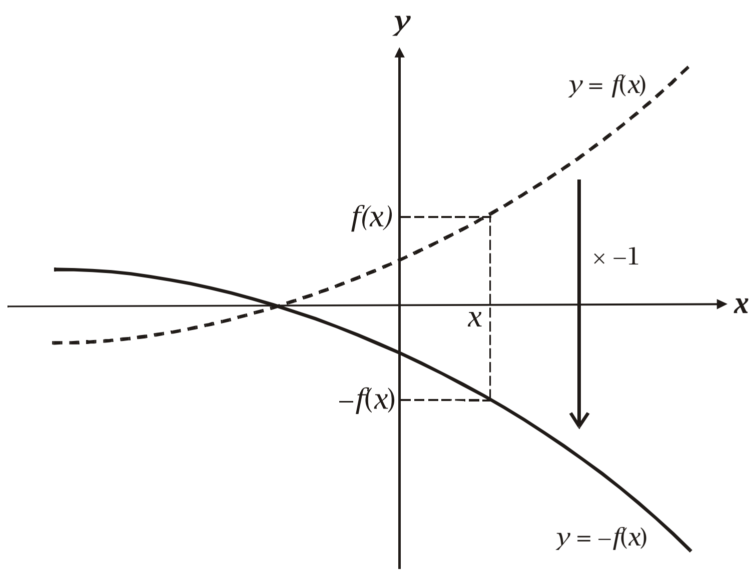


Vertical Scaling

$y = af(x)$
is a scaling of $y = f(x)$ by a scale factor of magnitude a
$y \to ay$

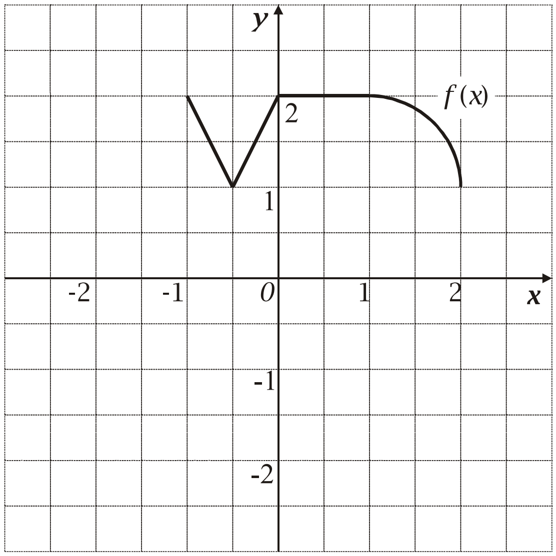


We multiply every value, y, by the scale factor a. If the scale factor a is negative this will reflect the graph of $y = f(x)$ in the x-axis and scale it.



Example (4)

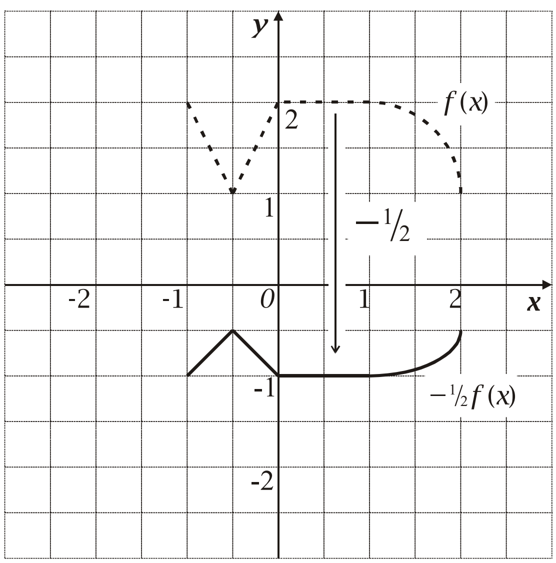
The function $y = f(x)$ has graph



What is the effect on this graph of the scaling $y \to -\frac{1}{2}y$?

Solution

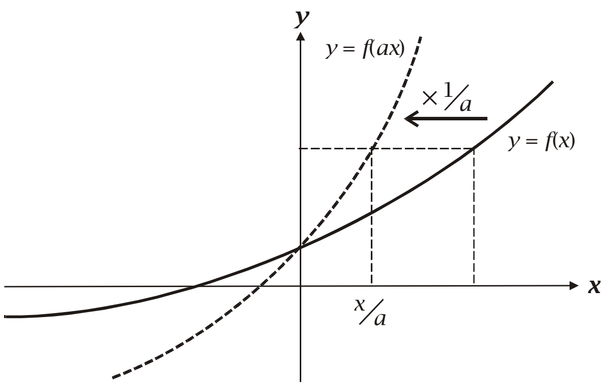
Every point on the graph is multiplied by $-\frac{1}{2}$. We obtain



Horizontal Scaling

$y = f(ax)$

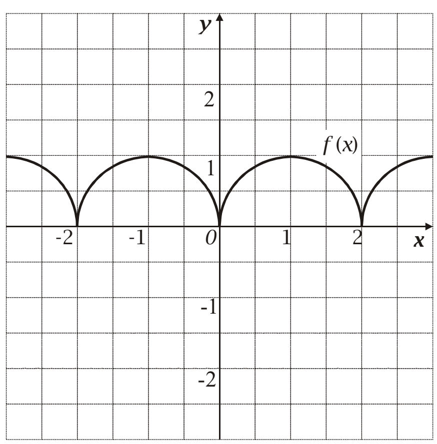
is a scaling of $y = f(x)$ by $1/a$ in the x-direction.



Since we are multiplying the x term by the factor a, we may wonder why the graph is not stretched rather than shrunk. However, the expression $f(ax)$ will take the same value as $f(x)$ at $x = x/a$ since $a \times x/a = x$. So multiplying x by the factor a inside the bracket is equivalent to shrinking the graph of $f(x)$ by a scale factor of $1/a$. If we wish to stretch $f(x)$ by the factor a, then we require the function $f\left(x/a\right)$.

Example (5)

The function $y = f(x)$ has the following graph



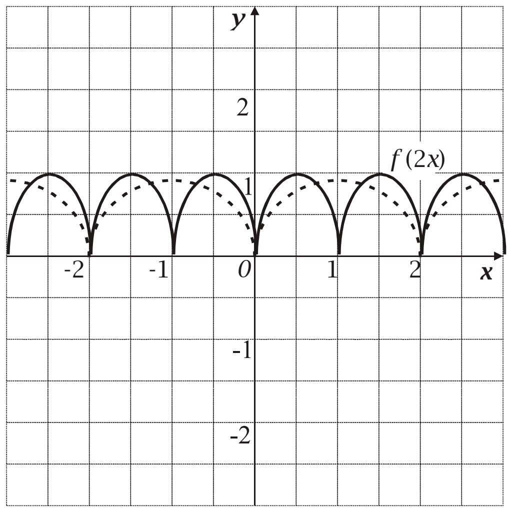
Make separate copies of this graph and sketch the functions

(a) $y = f(2x)$

(b) $y = f\left(\frac{1}{2}x\right)$

Solution

(a) $y = f(2x)$ is going twice as fast as $y = f(x)$, so the graph is shrunk by a scale factor of $\frac{1}{2}$. Its graph is



(b) $y = f\left(\frac{1}{2}x\right)$ is going twice as slow as $y = f(x)$, so the graph is increased by a scale factor of 2. Its graph is