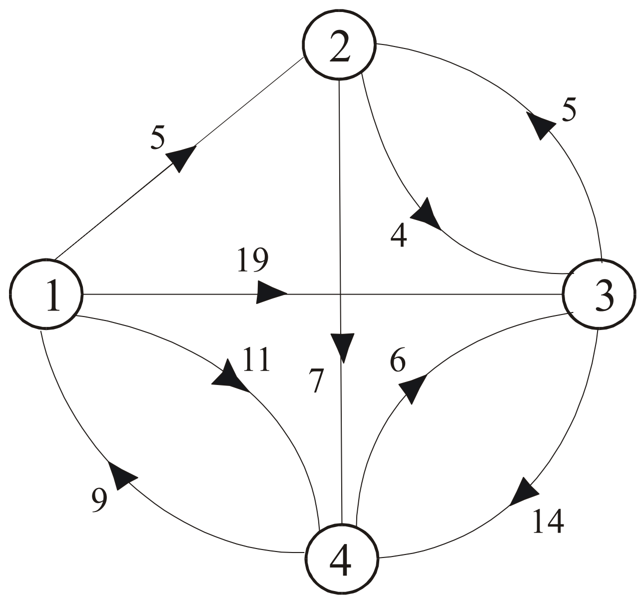
The Hungarian algorithm and the Travelling and Optimal Salesperson Problems

The travelling salesperson problem

Suppose we are given a weighted diagraph – that is a graph of vertices connected by directed edges each having a weight, which may be, for instance, the shortest distance between the two vertices.

For example, consider this diagraph.



The travelling salesperson is the problem of finding a closed path of minimal weight that passes through every vertex exactly once.

This problem is solved by using an algorithm that employs the Hungarian algorithm as a sub-routine.

Firstly, we find the weight matrix corresponding to the weighted diagraph. For the above example.

$$W = \begin{pmatrix} 0 & 5 & 19 & 11 \\ - & 0 & 4 & 7 \\ - & 5 & 0 & 14 \\ 9 & - & 6 & 0 \end{pmatrix}$$

In this matrix the *i,j*th entry corresponds to the edge connecting the *i*th vertex to the *j*th . For example

$$W = \begin{pmatrix} 0 & 5 & 19 & 11 \\ - & 0 & \boxed{\mathbf{4}} & 7 \\ - & 5 & 0 & 14 \\ 9 & - & 6 & 0 \end{pmatrix}$$

This corresponds to the edge connecting vertex (2) to vertex (3) in that direction. You should refer back to the original diagraph for confirmation of the correctness of the entry.

In this weight matrix note two points

(1) Diagonal elements, for example, the edge from vertex (1) to vertex (1) has 0 weight and hence a 0 entry. This is because there is no distance between a vertex and itself.

(2) When an edge does not exist it is represented by a dash (–). This means the entry is undefined.

From the weight matrix W we define another weight matrix A by replacing every 0 and undefined entry in W by an ∞ entry in A

Thus from W above we obtain

$$A = \begin{pmatrix} \infty & 5 & 19 & 11 \\ \infty & \infty & 4 & 7 \\ \infty & 5 & 0 & 14 \\ 9 & \infty & 6 & \infty \end{pmatrix}$$

The point of this is that when we subtract any number from ∞ we still obtain ∞; ie $\infty - 6 = \infty$. In other words the zero and undefined entries are deleted from the problem. However the symbol ∞ is cumbersome and obscured the progress of the subsequent algorithm; hence I recommend that it be replaced by a dot (·). We have to remember that in future processes a dot (·) remains a dot (·) and it is not modified by operations on rows or columns.

To begin solving the travelling salesperson problem we now apply the Hungarian algorithm to the matrix A. This means (1) reduce rows; (2) reduce columns ;(3) augment the matrix if necessary, and repeatedly, if necessary (4) deleting zeroes; (5) selecting entries and referring back to the original problem.

In our example

$$A = \begin{bmatrix} \cdot & 5 & 19 & 11 \\ \cdot & \cdot & 4 & 7 \\ \cdot & 5 & \cdot & 14 \\ 9 & \cdot & 6 & \cdot \end{bmatrix}$$

$\downarrow$ row reductions

$$A = \begin{bmatrix} \cdot & 0 & 14 & 6 \\ \cdot & \cdot & 0 & 3 \\ \cdot & 0 & \cdot & 9 \\ 5 & \cdot & 2 & \cdot \end{bmatrix}$$

$\downarrow$ column reductions

$$A = \begin{bmatrix} \cdot & 0 & 12 & 3 \\ \cdot & \cdot & 0 & 0 \\ \cdot & 0 & \cdot & 6 \\ 0 & \cdot & 2 & \cdot \end{bmatrix}$$

$\downarrow$ deleting zeroes

$$A = \begin{bmatrix} \cdot & \boxed{0} & 12 & 3 \\ \cdot & \cdot & \boxed{0} & 0 \\ \cdot & 0 & \cdot & 6 \\ \boxed{0} & \cdot & 2 & \cdot \end{bmatrix}$$

$\downarrow$ Does not have $n = 4$ solutions; hence augment matrix

$$A = \begin{array}{c c} \begin{bmatrix} \cdot & 0 & 12 & 3 \\ \cdot & \cdot & 0 & 0 \\ \cdot & 0 & \cdot & 6 \\ 0 & \cdot & 2 & \cdot \end{bmatrix} & \begin{array}{c} -3 \\ \\ -3 \\ \\ \end{array} \\ \begin{array}{cccc} & +3 & & \end{array} & \end{array}$$

$\downarrow$ subtract 4 from each uncovered row
add 4 to each covered column.

$$A = \begin{bmatrix} \cdot & 0 & 9 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & 0 & \cdot & 3 \\ 0 & \cdot & 2 & \cdot \end{bmatrix}$$

$\downarrow$ deleting zeroes

$$A = \begin{bmatrix} \cdot & 0 & 9 & \boxed{0} \\ \cdot & \cdot & \boxed{0} & 0 \\ \cdot & \boxed{0} & \cdot & 3 \\ \boxed{0} & \cdot & 2 & \cdot \end{bmatrix}$$

$\downarrow$

This solution is $1 \to 4$ with weight 11
$2 \to 3$ with weight 4
$3 \to 2$ with weight 5
$4 \to 1$ with weight 9

This has total weight 11 + 4 + 5 + 9 = 29

If this is a closed path connecting all the vertices then we have solved the travelling salesperson problem. However, there is, in general, no guarantee that the first application of the Hungarian algorithm will solve this problem because it may create a solution that involves sub tours. This is the case in our example, since we see that the algorithm has created a solution involving two sub tours which are not connected.

$1 \to 4 \to 1$
and $2 \to 3 \to 2$

We must modify the matrix A so that we are forced to connect the two sub tours to each other.

By way of passing note, the total weight of our first solution is 11+ 4 + 5 + 9 = 29. This provides a lower bound for the solution to travelling salesman problem – i.e. any solution will have total weight ≥ 29.

The first use of the algorithm has divided the set of vertices into two subsets {1,4 } and {2,3}. It has partitioned the set into two sub-sets.

We take the smallest of these sets and for each vertex in this set create modified weight matrices from the original weight matrix by deleting the entry from that vertex from successively each of the other vertices in the set. We delete only one entry at a time.

For example, we have sets {1,4} and {2,3}which both have the same size (order), which is two. Either can be selected , so we choose the set {1,4}. This contains the vertices 1 and 4. Taking the vertex 1, in the modified weight matrix A we therefore (1) delete the 1,4 entry to obtain another matrix ; (2) we separately delete the 4,1 entry to obtain a second matrix.

We repeat the process for the vertex 4; however, in this case, deleting 4,1 and deleting 1,4 results in the same two modified matrices. Thus in this example , we arrive at two further modified matrices.

$$A = \begin{bmatrix} . & 5 & 19 & 11 \\ . & . & 4 & 7 \\ . & 5 & . & 14 \\ 9 & . & 6 & . \end{bmatrix}$$

$$A_1 = \begin{bmatrix} . & 5 & 19 & . \\ . & . & 4 & 7 \\ . & 5 & . & 14 \\ 9 & . & 6 & . \end{bmatrix} \qquad A_2 = \begin{bmatrix} . & 5 & 19 & 11 \\ . & . & 4 & 7 \\ . & 5 & . & 14 \\ . & . & 6 & . \end{bmatrix}$$

We must now apply the Hungarian algorithm to each of the modified matrices A_1 and A_2. The modified matrices are called branching matrices. For each branching matrix the process of applying the Hungarian algorithm and possibly further modifying the matrix to obtain further branching matrices must be applied until either a feasible solution to the travelling salesperson problem is obtained or every possible avenue is exhausted without solution. At this point the best of the feasible solutions is selected, and that will be the solution to the travelling salesperson problem!

In our example we will start with the matrix A_2. Recall that the dots are shorthand for ∞ signs , so the branching matrix A_2 is really.

$$A_2 = \begin{bmatrix} \infty & 5 & 19 & 11 \\ \infty & \infty & 4 & 7 \\ \infty & 5 & \infty & 14 \\ \infty & \infty & 6 & \infty \end{bmatrix}$$

Looking at the first column, we see that it has only ∞ entries. Hence no amount of row or column operations will ever introduce a zero into this matrix. Therefore there is no solution to the travelling salesperson problem involving this branching matrix.

We now turn to the branching matrix A_1 and apply the Hungarian algorithm to it.

$$A_1 = \begin{bmatrix} . & 5 & 19 & . \\ . & . & 4 & 7 \\ . & 5 & . & 14 \\ 9 & . & 6 & . \end{bmatrix}$$

$\downarrow$ row reduction

$$A_1 = \begin{bmatrix} . & 0 & 9 & . \\ . & . & 0 & 3 \\ . & 0 & . & 9 \\ 3 & . & 0 & . \end{bmatrix}$$

$\downarrow$ column reduction

$$A_1 = \begin{bmatrix} . & 0 & 9 & . \\ . & . & 0 & 0 \\ . & 0 & . & 6 \\ 0 & . & 0 & . \end{bmatrix}$$

$\downarrow$ deleting zeroes

$$A_1 = \begin{bmatrix} . & \boxed{0} & 9 & . \\ . & . & 0 & \boxed{0} \\ . & 0 & . & 6 \\ \boxed{0} & . & 0 & . \end{bmatrix}$$

$\downarrow$ Only n = 3 solutions
augment the matrix

$$A_1 = \begin{array}{cc} \begin{bmatrix} . & 0 & 9 & . \\ . & . & 0 & 0 \\ . & 0 & . & 6 \\ 0 & . & 0 & . \end{bmatrix} & \begin{matrix} -6 \\ \\ -6 \\ \\ \end{matrix} \\ \begin{matrix} \quad & +6 & \quad & \quad \end{matrix} & \end{array}$$

$\downarrow$ -6 from each uncovered row
+6 to each covered column

$$A_1 = \begin{bmatrix} . & 0 & 3 & . \\ . & . & 0 & 0 \\ . & 0 & . & 0 \\ 0 & . & 0 & . \end{bmatrix}$$

$\downarrow$ delete zeroes

$$A_1 = \begin{bmatrix} . & \boxed{0} & 3 & . \\ . & . & \boxed{0} & 0 \\ . & 0 & . & \boxed{0} \\ \boxed{0} & . & 0 & . \end{bmatrix}$$

$\downarrow$

Solution	$1 \rightarrow 2$	5
	$2 \rightarrow 3$	4
	$3 \rightarrow 4$	14
	$4 \rightarrow 1$	9

Total weight is 5 + 4 + 14 + 9 = 32

This defines a closed path through each vertex and is, therefore, a feasible solution to the travelling salesperson problem. Since we have exhausted the feasible solutions this is the only solution and hence the solution to the original problem.

Some further terminology.

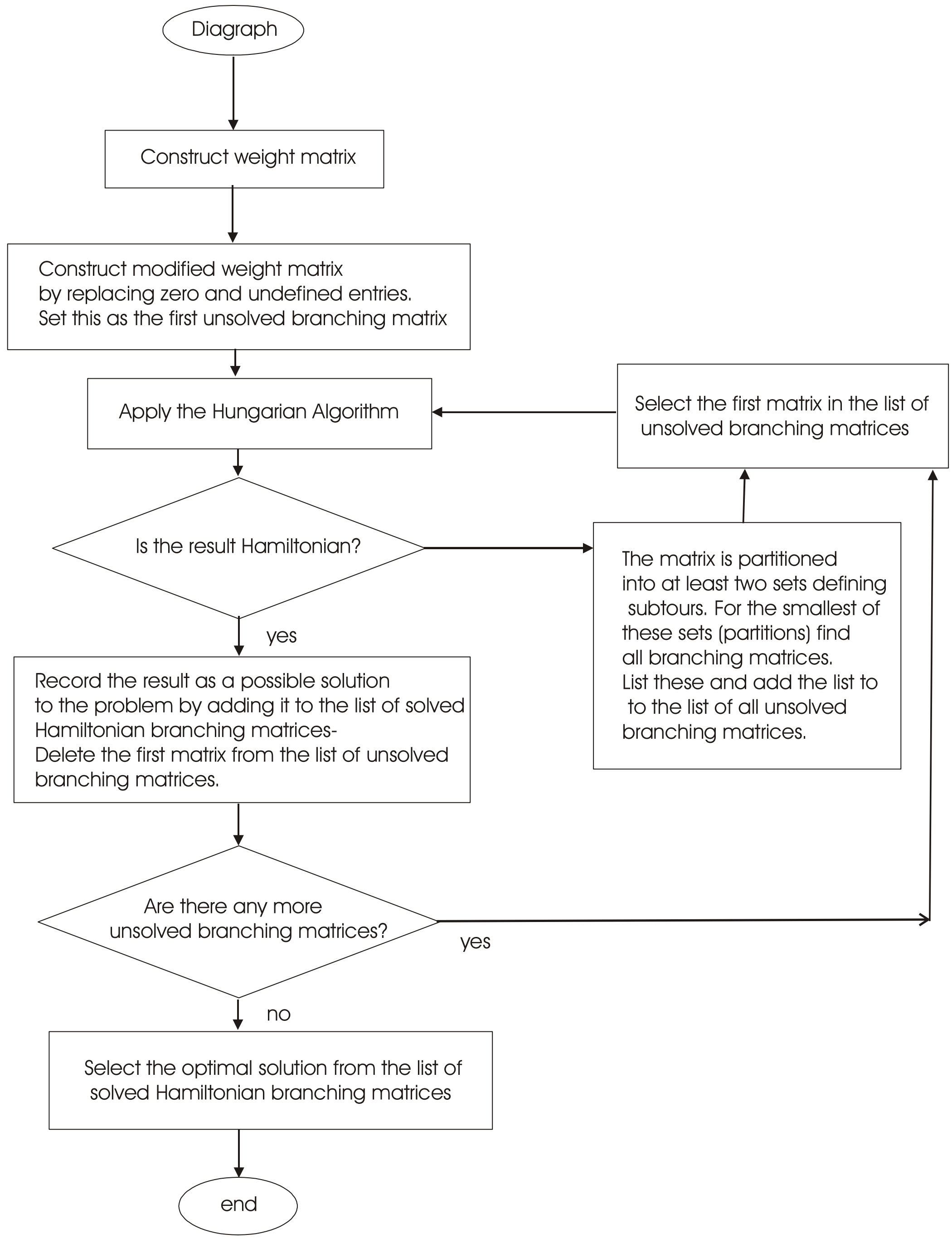
A closed path that passes through every vertex of a diagraph exactly once is called a Hamiltonian cycle.

The travelling salesperson problem is, therefore, also called the optimal Hamiltonian problem – The problem of finding a Hamiltonian cycle of minimum weight.

A matrix will be called a Hamiltonian matrix if a single application of the Hungarian algorithm yields a Hamiltonian cycle (that is, a feasible solution to the travelling salesperson problem; meaning a possible solution without subtours.)

An algorithm for the travelling salesperson problem follows on the next page.

Algorithm for the travelling salesperson problem.



Optimal salesperson problem

Whereas the travelling salesperson problem is the problem of finding a closed path of minimal weight that passes through every vertex exactly once, the optimal salesperson is the problem of finding a closed path of minimal weight that passes through every vertex at least once.

To solve the optimal salesperson problem, solve the travelling salesperson problem. Then examine each connection between vertices in the optimal solution to the travelling salesperson problem. If there is a shorter path between two vertices than the direct connection between them, replace that direction connection by the shortest such alternative path. When all connections between vertices have been examined you will have the solution to the optimal salesperson problem.

For example, the solution to the travelling salesperson problem for the matrix

$$\begin{matrix} \infty & 5 & 9 & 11 \\ 16 & \infty & 4 & 7 \\ 21 & 5 & \infty & 12 \\ 9 & 11 & 6 & \infty \end{matrix}$$

is the path $1 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$ with weight 9 + 5 + 7 + 9 = 30. But there is a shorter path from 3 to 4, which is $3 \rightarrow 2 \rightarrow 4$. Therefore a solution to the optimal salesperson problem is $1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 4 \rightarrow 1$.