The Travelling Salesperson Problem

The travelling and optimal salesperson problems

Once upon a time there was a travelling salesperson. Travelling salespeople have to undergo long journeys through many towns. They get tired. They would like to find the shortest route that takes them through every town once and only once. This is the travelling salesperson problem.

For a given arrangement of towns there will be a solution to the travelling salesperson problem provided the towns are connected. The optimum solution to the travelling salesperson problem can found by an algorithm that involves the use of the Hungarian algorithm to solve a matching problem. So the travelling salesperson problem has been solved.

Thus the travelling salesperson problem, if it has a solution, can be completely solved by the use of the Hungarian algorithm. However, as the Hungarian algorithm involves a "step up", so to speak, in terms of mathematical sophistication, it is usual to begin the study of this problem as an application of the minimum connector problem.

You are reminded that the minimum connector problem is the problem of finding the minimum spanning tree of a graph. A graph involves connections – edges (or arcs) – between vertices. There will be a graph derived from any graph that connects all the vertices by the minimum number of edges – this is called the minimum spanning tree.

We introduce the use of the minimum spanning tree to approach the travelling salesperson problem by means of a worked example.

Example

The following matrix represents the distances in metres between sites of interest A to G in a park

$$\begin{array}{cc}
 & \begin{array}{ccccccc} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} & \mathrm{F} & \mathrm{G} \end{array} \\
\begin{array}{c} \mathrm{A} \\ \mathrm{B} \\ \mathrm{C} \\ \mathrm{D} \\ \mathrm{E} \\ \mathrm{F} \\ \mathrm{G} \end{array} &
\begin{pmatrix}
- & 5 & - & 3 & 8 & - & 6 \\
5 & - & 7 & - & - & 4 & 4 \\
- & 7 & - & 9 & 2 & 10 & 15 \\
3 & - & 9 & - & 6 & - & - \\
8 & - & 2 & 6 & - & 7 & - \\
- & 4 & 10 & - & 7 & - & - \\
6 & 4 & 15 & - & - & - & -
\end{pmatrix}
\end{array}$$

Jonny is planning a party in the park in which there will be a treasure hunt. He wants to check that every site of interest would be suitable for a clue in the treasure hunt so he needs to visit all the sites. He starts from *A*, which is the park gate. He doesn't want to walk a lot, so he wishes to vist very site by walking the minimum distance.

(a) By using Prim's algorithm find a lower bound for the minimum distance.

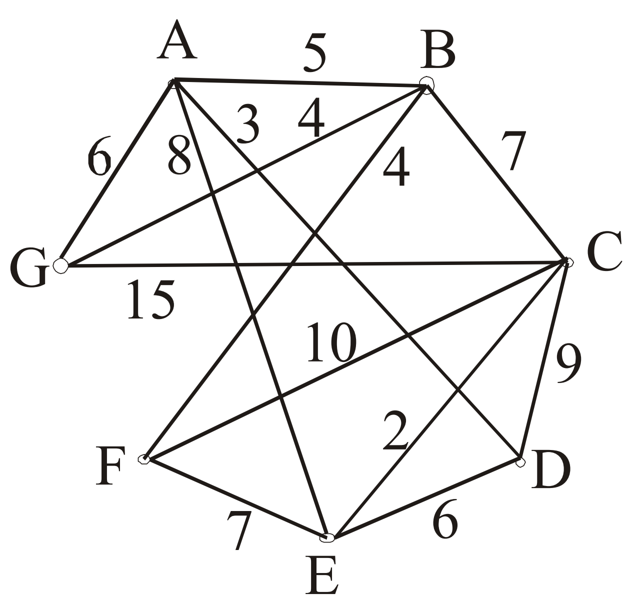
(b) Let *d* be the distance obtained in part (a) for the lower bound for the minimum distance. Explain why an upport bound for the minimum distnce is 2*d*.

By consider the fact that Jonny need not end his tour of the park at *A* where he starts, obtaining an improved route for the minimum distance with a length not greater than 37 meteres.

Shown how this upper bound can be reduced to 34 metres by the introduction of one cycle – that is the inclusion of at least on edge that is not contained in the minimum spanning tree.

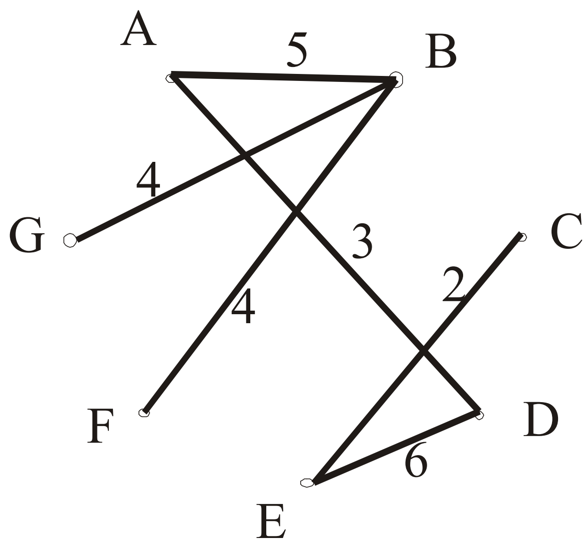
Solution

We begin by drawing a graph corresponding to the matrix representation of the topology of the park.



(a) We shall find a minimum connector of the graph shown above using Prim's algorithm.

Thus, using Prim's algorithm, starting a vertex *A*, the minimum connector is



The edges are added in the following order.

Edge	Length
AD	3
AB	5
BF	4
BG	4
DE	6
EC	2

This is a minimum spanning tree, so it must be a lower bound for Johnny's walk. The minimum distance is not lower than 24 metres. It is not posiible to visit each node by a shorter route.

(b) We start with the result of part (a). The first upper bound is twice of the minimum spanning tree. This would be the route if Johnny went along each arc of this graph twice - backwards and forwards. This would be the route

ABGBFGA ADECEDA

This route is 48 metres long.

But we can improve upon this upper bound of minimal distance.

Johnny starts from A, but he have to finish at A too. Let us look the spanning tree graph. Two arcs start from A – so Jonny will need to go back to A once. The longest distance from A is the arc C so it would be a good idea to finish his tour of the park by ending at C.

At B there are three edges in the minimum spanning tree, so Jonny must go back to B and continue along other path

So, the second upper bound is the route

ABFBGBA DEC

This is 37 metres long.

If Johnny go back to A from G but not by the GBA route, but by the GA direct route, then the total length is shorter. This introduces one cycle into his journey and the edge GA which is not in the minimumspanning tree.

However, should adopt the route

ABFBGA DEC

which is 34 metres long, this would constitute a new upper bound for he minimum distance.

So the exploration of the travelling salesperson problem via the use of the minimum spanning tree involves establishing both a lower and upper bound using the spanning tree, and then working *by trial and error* for an improved solution by the elimination of unnecessarily costly routes.