Trigonometric Equations

Prerequisites

You should be familiar with (1) the basic trigonometric identities, including the compound and double angle formulae; (2) De Moivre's theorem and its applications to trigonometry.

Trigonometric identities

Here is a summary of the trigonometric identities that you should know.

Basic identities

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin^2 A + \cos^2 A = 1$$

$$\tan^2 A + 1 = \sec^2 A$$

Odd and even functions

$$\sin(-A) = -\sin A$$

$$\cos(-A) = \cos A$$

$$\tan(-A) = -\tan A$$

Compound angle formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Double angle formulae

$$\sin 2A = 2\sin A\cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A\end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

$$\cos^2 A = \frac{\cos 2A + 1}{2}$$

Example (1)

Use the double angle formulae to prove: $\cos 3A = 4\cos^3 A - 3\cos A$

Solution

$$\begin{aligned}\cos 3A &= \cos(2A \cdot A)\\ &= \cos 2A\cos A - \sin 2A \sin A\\ &= \left(2\cos^2 A - 1\right)\cos A - \left(2\sin A\cos A\right)\sin A\\ &= 2\cos^3 A - \cos A - 2\left(1-\cos^2 A\right)\cos A\\ &= 4\cos^3 A - 3\cos A\end{aligned}$$

De Moivre's theorem

In Cartesian form this is

$$z^n = r(\cos\theta + i\sin\theta)^n = r^n\left(\cos n\theta + i\sin n\theta\right)$$

and in polar form

$$z^n = [r,\theta]^n = [r^n, n\theta]$$

Example (2)

Use De Moivre's theorem to prove: $\sin 3A = 3\sin A - 4\sin^3 A$

Solution

By De Moivre's theorem

$$\begin{aligned}\left(\cos 3A + i\sin 3A\right) &= \left(\cos A + i\sin A\right)^3\\ &= \cos^3 A + 3i\cos^2 A\sin A + 3i^2\cos A\sin^2 A + i^3\sin^3 A\\ &= \cos^3 A - 3\cos A\left(1-\cos^2 A\right) + i\left(3\sin A\left(1-\sin^2 A\right) - \sin^3 A\right)\\ &= 4\cos^3 A - 3\cos A + i\left(3\sin A - 4\sin^3 A\right)\end{aligned}$$

On equating imaginary parts

$$\sin 3A = 3\sin A - 4\sin^3 A$$

Equating real parts also gives us the result we proved in example (1)

$$\cos 3A = 4\cos^3 A - 3\cos A$$

Solving trigonometric equations

We can apply the above formulae to solve trigonometric equations.

Example (3)

Solve the equation $\cos A + \cos 2A + \cos 3A = 0$

Solution

Substituting into this equation from the identities

$\cos 2A = 2\cos^2 A - 1$

$\cos 3A = 4\cos^3 A - 3\cos A$

we get

$\cos A + \left(2\cos^2 A - 1\right) + \left(4\cos^3 A - 3\cos A\right) = 0$

$4\cos^3 A + 2\cos^2 A - 2\cos A - 1 = 0$

Let $t = \cos A$

$4t^3 + 2t^2 - 2t - 1 = 0$

So we need to find the factors of

$f(t) = 4t^3 + 2t^2 - 2t - 1$

Using the remainder theorem to search for a factor

$f(0) = -1$

$f(1) = 4 + 2 - 2 - 1 = 3$

$f(-1) = -4 + 2 + 2 - 1 = -1$

$f\left(-\frac{1}{2}\right) = -\frac{4}{8} + \frac{2}{4} + 1 - 1 = 0$

So $\left(t + \frac{1}{2}\right)$ or $(2t + 1)$ is a factor.

By polynomial division

$$\begin{array}{r|l} & 2t^2 - 1 \\ 2t + 1 & 4t^3 + 2t^2 - 2t - 1 \\ & \underline{4t^3 + 2t^2} \\ & \quad\quad -2t - 1 \\ & \quad\quad \underline{-2t - 1} \end{array}$$

Hence

$f(t) = 4t^3 + 2t^2 - 2t - 1 = (2t + 1)\left(\sqrt{2}t - 1\right)\left(\sqrt{2}t + 1\right)$

Whence

$4t^3 + 2t^2 - 2t - 1 = 0 \qquad\qquad t = \cos A$

implies

$\cos A = -\frac{1}{2}$ $\quad \cos A = \frac{1}{\sqrt{2}}$ $\quad \cos A = -\frac{1}{\sqrt{2}}$

and

$A = 2n\pi \pm \frac{2\pi}{3}$ $\quad A = 2n\pi \pm \frac{\pi}{4}$ $\quad A = 2n\pi \pm \frac{3\pi}{4}$ $\quad n = 0, \pm 1, \pm 2, \ldots$

The substitution $t = \tan\left(\frac{x}{2}\right)$

The substitution $t = \tan\left(\frac{x}{2}\right)$ can be used to solve certain trigonometric equations. Firstly, we will show that the substitution $t = \tan\left(\frac{x}{2}\right)$ gives

$\tan x = \frac{2t}{1-t^2}$ $\quad \sin x = \frac{2t}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$

To show this we start with the trigonometric identity

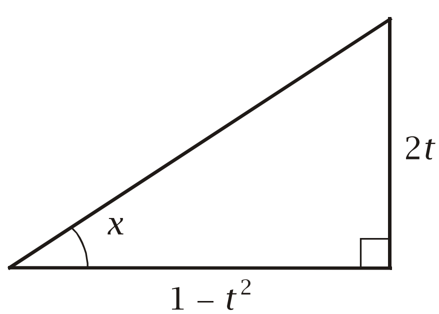
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Substituting $A = \frac{x}{2}$ into this gives

$$\tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$

$$\tan x = \frac{2t}{1-t^2}$$

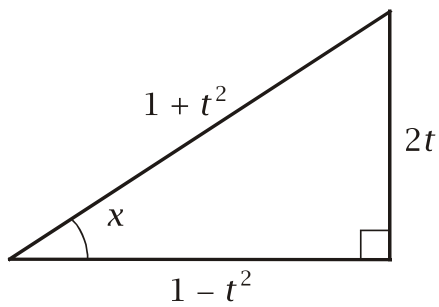
So, if we interpret x as an angle, this gives us the triangle



The hypotenuse of this triangle is found by Pythagoras's theorem

$$\text{hypotenuse} = \sqrt{(2t)^2 + (1+t^2)^2}$$
$$= \sqrt{4t^2 + 1 - 2t^2 + t^4}$$
$$= \sqrt{t^4 + 2t + 1}$$
$$= \sqrt{(t^2+1)^2}$$
$$= t^2 + 1$$

Hence



From this we can deduce that

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Example (4)

Solve the equation: $2\sin x - \tan\left(x/2\right) = 0$

Solution

Substituting $t = \tan\left(\frac{x}{2}\right)$ and $\sin x = \frac{2t}{1+t^2}$ into $2\sin x - \tan\left(x/2\right) = 0$ gives

$$2\left(\frac{2t}{1-t^2}\right) - t = 0$$
$$4t - t\left(1-t^2\right) = 0$$
$$t^3 - 3t = 0$$
$$t\left(t+\sqrt{3}\right)\left(t-\sqrt{3}\right) = 0$$

$t = 0$	$t = -\sqrt{3}$	$t = \sqrt{3}$	
$\tan\left(\frac{x}{2}\right) = 0$	$\tan\left(\frac{x}{2}\right) = -\sqrt{3}$	$\tan\left(\frac{x}{2}\right) = \sqrt{3}$	
$\frac{x}{2} = n\pi$	$\frac{x}{2} = n\pi + \frac{\pi}{3}$	$\frac{x}{2} = n\pi - \frac{\pi}{3}$	$n = 0, \pm 1, \pm 2, \ldots$
$x = 2n\pi$	$x = 2n\pi + \frac{2\pi}{3}$	$x = 2n\pi - \frac{2\pi}{3}$	$n = 0, \pm 1, \pm 2, \ldots$