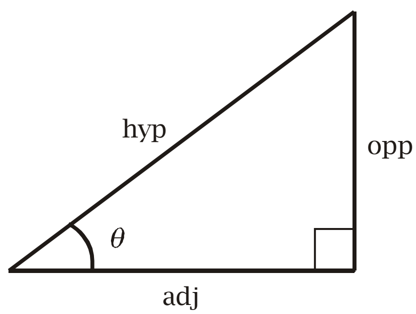
Trigonometric Functions

Prerequisites

The essential background for this topic is (1) familiarity with trigonometric ratios, (2) working knowledge of the concept of a function.

(1) Trigonometric ratios

The following diagram represents a right-angled triangle.



The side next to the angle *x* is called the *adjacent* side, abbreviated to *adj*; the side opposite *x* is called the *opposite* side (*opp*); and the longest side that is opposite the right angle is called the *hypotenuse* (*hyp*). The trigonometric ratios are

$$\sin\theta = \frac{opp}{hyp} \qquad \cos\theta = \frac{adj}{hyp} \qquad \tan\theta = \frac{opp}{adj}$$

where sin, cos and tan are abbreviations for sine, cosine and tangent respectively.

(2) Functions

A function is a rule that takes you from one number to another. Other equivalent terms for a function are *map* or *mapping*. There are various ways of writing functions in mathematics. For example, the *mapping diagram*

$$f: \quad x \to y$$

describes the function *f* as a rule (or mapping) of one number *x* to another number *y*. We say "*f* maps *x* to *y*". This is also written

$$y = f(x)$$

We say "*y* is *f* of *x*" – *y* is the *value* of the function *f* when *f* takes the *argument x*.

Example (1)

Let *f* be the function $f(x) = 2x^3 + 3$. Find $f(-2)$.

Solution

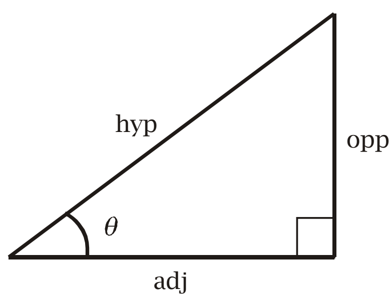
$f(-2) = 2 \times (-2)^3 + 3 = -13$

We can also write this using a mapping diagram as

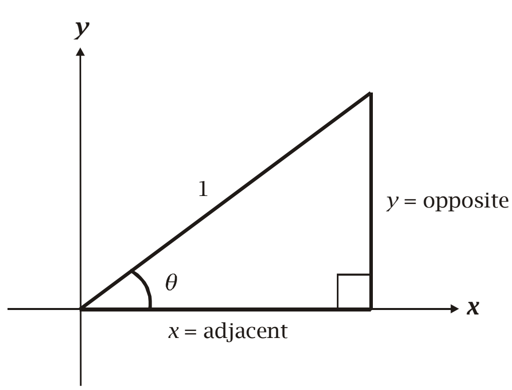
$f: \quad -2 \to -13$

Trigonometric functions

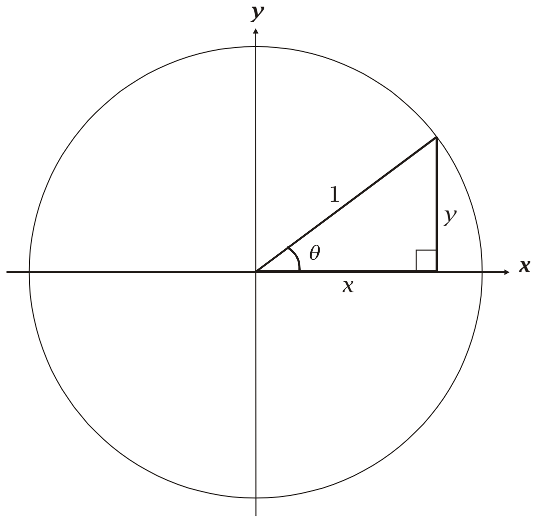
We will extend the concept of a trigonometric ratio and turn it into a function. The trigonometric ratios are defined for the following triangle.



Suppose we transpose this diagram onto a set of coordinate axes, and at the same time, choose a hypotenuse of length 1 unit.



We have replaced the terms "adjacent" and "opposite" by variables *x* and *y*. We now inscribe the figure inside a circle.



The diagram shows that as the angle θ varies the lengths x and y also vary. This gives

$\cos\theta = \dfrac{adj}{hyp} = \dfrac{x}{1}$ $\qquad\qquad$ $\sin\theta = \dfrac{opp}{hyp} = \dfrac{y}{1}$

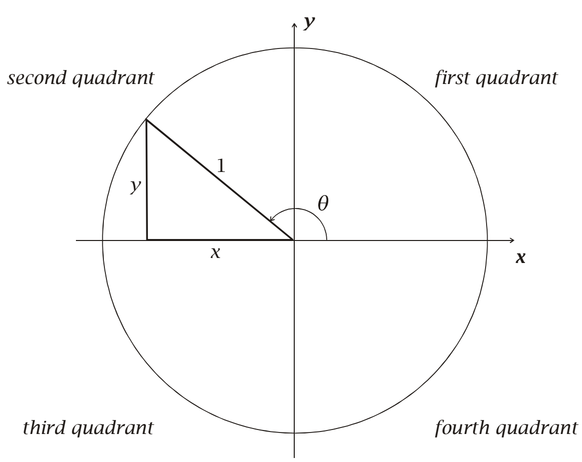
Hence

$x = \cos\theta$ $\qquad\qquad$ $y = \sin\theta$

The "lengths" x and y may be thought of as the images of the functions

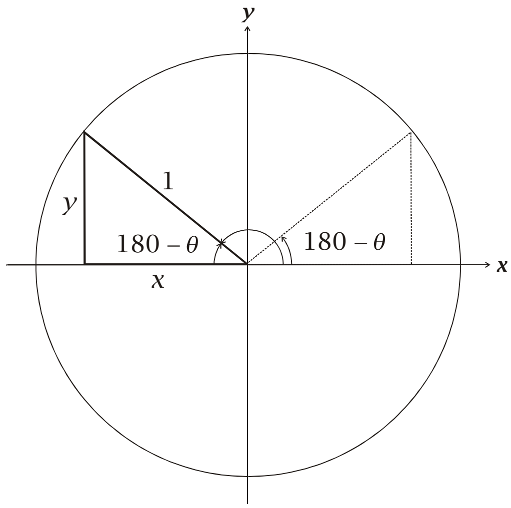
$f_x(\theta) = \cos\theta$ $\qquad\qquad$ $f_y(\theta) = \sin\theta$

respectively. These define the *trigonometric functions*, $\cos\theta$ and $\sin\theta$. They are mappings from the angle to the respective trigonometric ratio associated with that angle. Now consider what happens to $x = \cos\theta$ as the angle θ increases above 90°.



As it increases the angle θ passes through four *quadrants* of the diagram.

In the second quadrant the value of x becomes negative. The size of x is equal to the cosine of the angle $180-\theta$ as this diagram shows



So, when $90 < \theta \le 180$ we have $\cos\theta = -\cos(180-\theta)$. Letting $x = \cos\theta$ in the same way as θ varies throughout 360° we can obtain the following table of values.

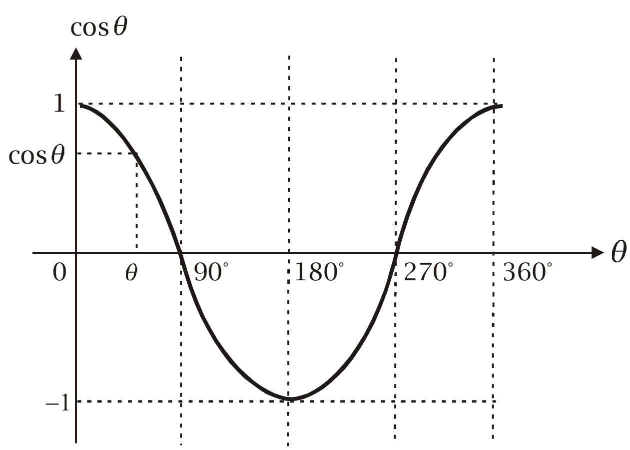
x	0	30°	45°	60°	90°	120°	135°	150°	180°
$\cos x$	1	0.87	0.71	0.5	0	–0.5	–0.71	–0.87	–1

x	210°	225°	240°	270°	300°	315°	330°	360°
$\cos x$	–0.87	–0.71	–0.5	0	0.5	0.71	0.87	1

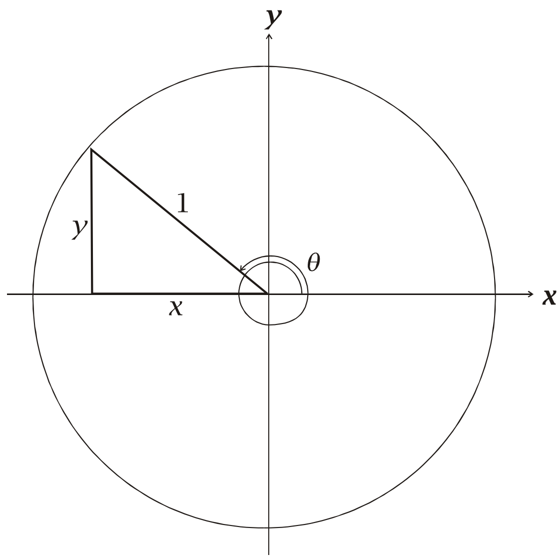
Example (2)

From the above table, plot a graph of $x = \cos\theta$ as a function of θ.

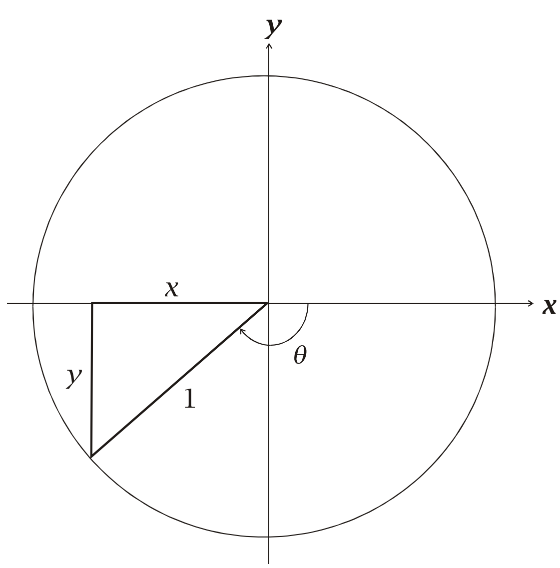
Solution



In this graph the value of $\cos\theta$ is defined for $0 < \theta \leq 180$; but why should we limit the values of $\cos\theta$ to this? If we carry on rotating the angle past 360° then we can continue to define $\cos\theta$ as the length $x = \cos\theta$ in the diagram

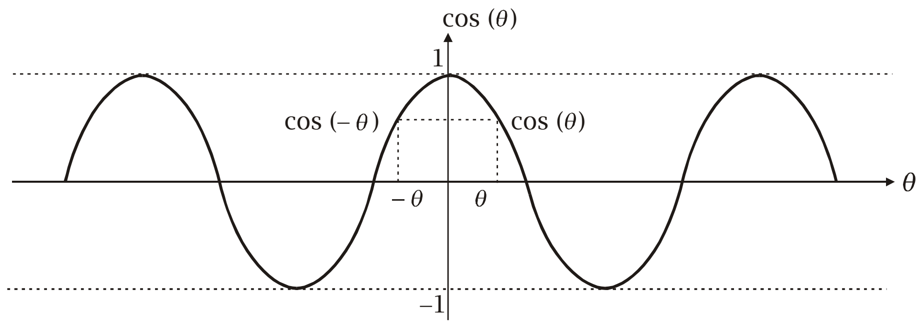


By reversing the direction in which the angle θ is measured, we can extend the graph to include negative values of θ.



By convention a positive value of θ is taken in an *anti-clockwise* direction from the positive *x*-axis; a negative value of θ is taken in a *clockwise* direction from the positive *x*-axis.

Thus, we obtain a graph of $x = \cos\theta$ for all values of θ.



This graph is *periodic*. This means that after every period of 360° the graph repeats itself.

$\cos\theta = \cos(\theta + n360°)$

where n is an integer, positive or negative. By similar arguments we can construct functions for

$\sin\theta : \theta \to \sin\theta$

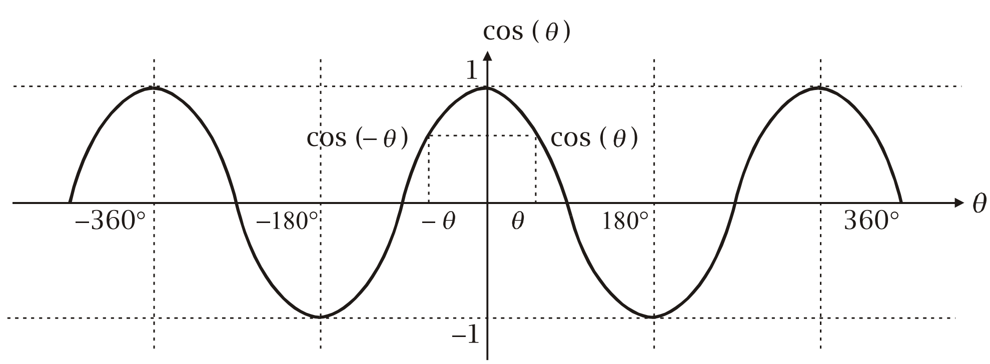
$\tan\theta : \theta \to \tan\theta$.

Graphs of trigonometric functions

You should be able to sketch the graphs of $\sin\theta$, $\cos\theta$ and $\tan\theta$ from memory.

(1) cosine

$\cos\theta : \theta \to \cos\theta$ is a *symmetric* or *even function*. This means that the graph is symmetric about the *y*-axis.

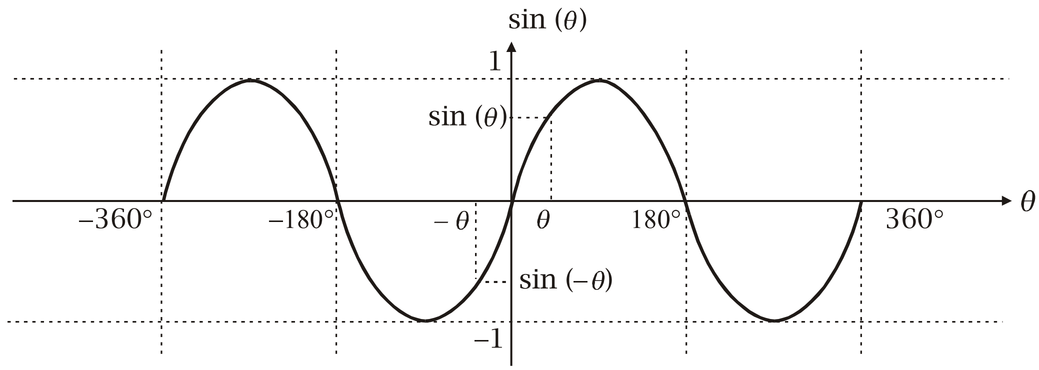


We also have the relationship

$\cos(-\theta) = \cos(\theta)$

(2) **sine**

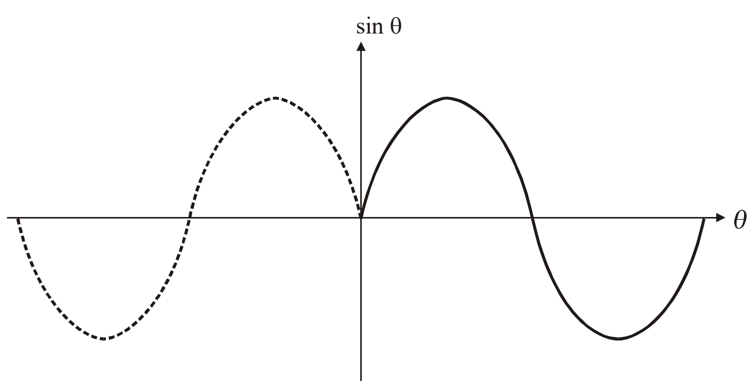
$\sin\theta : \theta \to \sin\theta$



This shows the graph of $y = \sin\theta$. The graph illustrates that $\sin\theta$ is periodic with period 360°. The graph also shows that

$\sin\theta = -\sin(-\theta)$

This is a sort of near symmetry. If the negative part of the graph were reflected in the θ-axis (the horizontal axis) the two graphs would become symmetric about the y-axis.

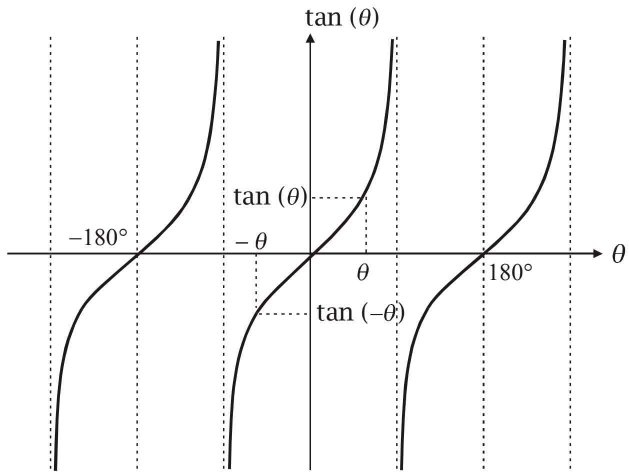


For this reason we say that $y = \sin\theta$ is an *anti-symmetric* function. We also call it an *odd* function, in contrast to functions that are purely symmetric or "even" functions.

(3) **tangent**

Another anti-symmetric, or odd, function is

$\tan\theta : \theta \to \tan\theta$



The graph would be symmetric if the negative part of it were reflected in the θ-axis. We also have

$\tan(-\theta) = -\tan(\theta)$

The graph illustrates that $\tan(\theta)$ is a periodic function with period 180°.

Example (3)

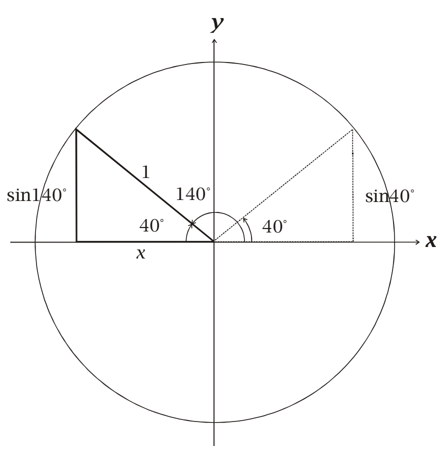
You are given $\sin 40° = 0.643$ (3.s.f.). Find

(*a*) $\sin 140°$ (*b*) $\sin 220°$ (*c*) $\sin(-40°)$ (*d*) $\sin 400°$

To solve these questions you use your knowledge of how the sign of $\sin\theta$ changes with θ and the fact that $\sin\theta$ is a periodic function, with period 360°

Solution

(*a*) $\sin 140°$ is equal in size to $\sin 40°$ and is positive.



$\sin 140° = \sin 40° = 0.643$ (3.s.f.)

(*b*) $220° = 180° + 40°$

In the third quadrant sine takes a negative sign. Thus

$\sin 220° = -\sin 40° = -0.643 \quad (3.\text{s.f.})$

(*c*) $\sin(-40°) = -\sin 40° = -0.643$

(*d*) $\sin\theta$ is periodic with period 360°.

$\sin 400° = \sin 40° = 0.643$