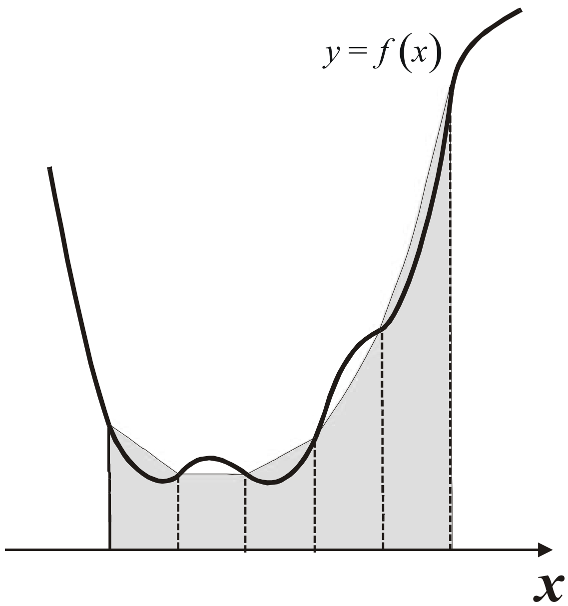
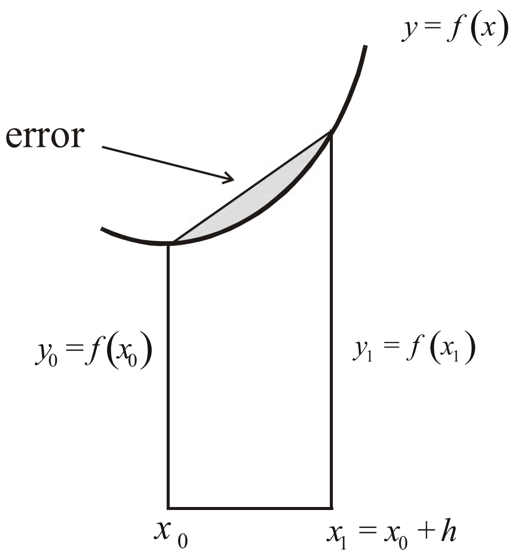
Truncation Errors

Errors in the trapezium rule

The trapezium rule estimates the area under the curve $f(x)$ by a series of trapezia, each of width h.



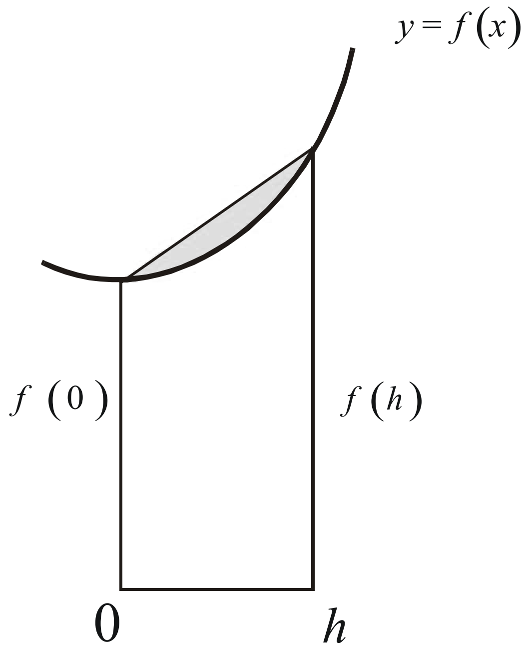
We wish to find a bound for the error created by using this approximation. Firstly, consider a single trapezium.



Let the true area under the curve be I. Then

$$I = \int_{x_0}^{x_0+h} f(x)\delta x$$

The area would be the same if the whole function were shifted to the left by x_0 - as shown here



The trapezium rule for this area gives

$$A = \frac{h}{2}\{f(0) + f(h)\}$$

But the real value is

$$I = \int_0^h f(x)\,\delta x$$

Having made this translation (the shift to the left by x_0) we can use the Taylor expansion for $f(x)$ about zero.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + ...$$

on substituting into (1)

$$I = \int_0^h \left(f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + ... \right) \delta x$$

$$= \left[xf(0) + \frac{x^2}{2} f'(0) + \frac{x^3 f''(0)}{3!} + ... \right]$$

$$= hf(0) + \frac{h^2}{2} f'(0) + \frac{1}{6} h^3 f''(0) + ...$$

$$= hf(0) + \frac{1}{2} h^2 f'(0) + \frac{1}{6} h^3 f''(0) + ...$$

The error in using the trapezium rule to this interval is

$$\text{error} = I - A$$

$$= \left\{ hf(0) + \frac{1}{2} h^2 f'(0) + \tfrac{1}{6} h^3 f''(0) + ... \right\} - \frac{h}{2} \{ f(0) + f(h) \}$$

Now the expression $f(h)$ can also be expanded as a Taylor series, giving

$$f(h) = f(0) + hf'(0) + \tfrac{h}{2} f''(0) + ...$$

which on substitution into (2) gives

$$\text{error} = I - A$$

$$= \{ hf(0) + \tfrac{1}{2} h^2 f'(0) + \tfrac{1}{6} h^3 f''(0) + ... \} - \tfrac{h}{2} (f(0) + f(0) + hf'(0) + \tfrac{h}{2} f''(0) + ...)$$

$$= \{ hf(0) + \tfrac{1}{2} h^2 f'(0) + \tfrac{1}{6} h^3 f''(0) + ... \} - \{ hf(0) + \tfrac{1}{2} h^2 f'(0) + \tfrac{1}{4} h^3 f''(0) + + ... \}$$

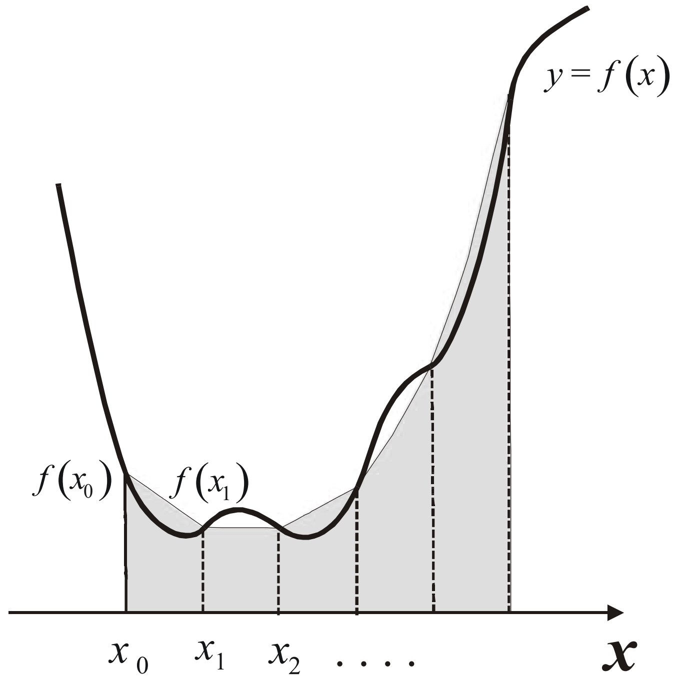
$$= \left(\tfrac{1}{6} - \tfrac{1}{4} \right) h^3 f''(0) + ...$$

$$= -\tfrac{1}{12} h^3 f''(0) + \text{higher degree terms}$$

So the principal error, called the truncation error, by using the trapezium rule to estimate a single strip is of the magnitude

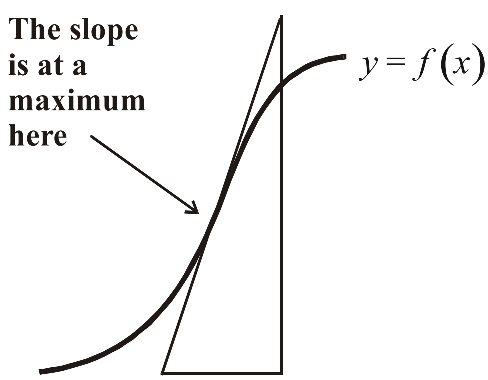
$\frac{1}{12}h^3 f''(0)$

If there are n strips then the principle error will be the sum of the errors of each strip.



$$\left|\text{error}\right| \approx \left|\tfrac{h^3}{12} f(x_0)\right| + \left|\tfrac{h^3}{12} f(x_1)\right| + \ldots + \left|\tfrac{h^3}{12} f(x_{n-1})\right|$$

Suppose $M =$ the maximum value of $\left|f'(x)\right|$ in the region over which the approximation is being taken.



Then $|\text{error}| \approx n \times \frac{h^3}{12} \times M$

If the interval is from a to b and the approximation involves taking n strips, of width h, then

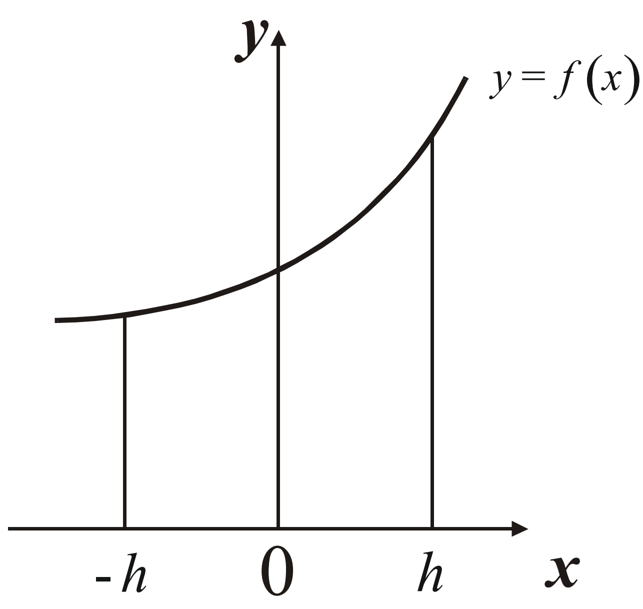
$$n = \frac{b-a}{h}$$

So the principal error in the trapezium rule can also be written

$$\begin{aligned}|\text{error}| &= \left(\frac{b-a}{h}\right)\frac{h^3}{12} \times M \\ &= (b-a)\frac{h^2}{12} \times M\end{aligned}$$

Error in Simpson's rule

We begin by considering the approximation to the area under a function, $y = f(x)$ when Simpson's rule is used with just to strips. We will also picture the function as centered about the origin and the strips will have width h.



The integral is

$$I=\int_{-h}^{h} f(x)\delta x$$

We will substitute the Taylor polynomial for $f(x)$ about zero to obtain

$$I=\int_{-h}^{h}\left\{f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f''''(0)+.....\right\}\delta x$$

$$=\left[xf(0)+\frac{x^2}{2}f'(0)+\frac{x^3}{3!}f''(0)+\frac{x^4}{4!}f'''(0)+\frac{x^5}{5!}f''''(0)+.....\right]_{-h}^{h}$$

$$=\left(hf(0)+\tfrac{h^2}{2}f'(0)+\tfrac{h^3}{3!}f''(0)+\tfrac{h^4}{4!}f'''(0)+\tfrac{h^5}{5!}f''''(0)+.....\right)$$

$$-\left(-hf(0)+\tfrac{h^2}{2}f'(0)-\tfrac{h^3}{3!}f''(0)+\tfrac{h^4}{4!}f''''(0)-\tfrac{h^5}{5!}f'''''(0)+.....\right)$$

$$=2hf(0)+2\tfrac{h^5}{5!}f''''(0)+.....$$

$$=2hf(0)+\tfrac{h^3}{3}f''(0)+\tfrac{h^5}{60}f''''(0)+....$$

Using Simpson's rule we can approximate the area under $f(x)$ by

$$A=\tfrac{h}{3}\left(f(-h)+4f(0)+f(h)\right)$$

We substitute for $f(-h)$ and $f(h)$ in this expression using Taylor polynomials

$$A=\tfrac{h}{3}\left\{\left(f(0)-hf'(0)+\tfrac{h^2}{2!}f''(0)+\tfrac{h^3}{3!}f'''(0)+\tfrac{h^4}{4!}f''''(0)+....\right)\right.$$

$$\left.+4f(0)+\left(f(0)+hf'(0)+hf''(0)+\tfrac{h^3}{3!}f'''(0)+\tfrac{h^4}{4!}f''''(0)+....\right)\right\}$$

$$=\tfrac{h}{3}\left\{6f(0)+2\tfrac{h^2}{2!}f''(0)+2\tfrac{h^3}{3!}f'''(0)+\tfrac{h^4}{4!}f''''(0)+....\right\}$$

$$=2hf(0)+\tfrac{h^3}{3}f''(0)+\tfrac{h^5}{36}f''''(0)+....$$

Then the principle error in the use of the Simpson's rule is

$$I-A=\left(\tfrac{1}{60}-\tfrac{1}{36}\right)\tfrac{h^5}{36}f''''(0)+\text{ higher degree terms}$$

$$=-\tfrac{1}{90}h^5f''''(0)+\text{ higher terms}$$

This because the first two terms in the expansion of $I + A$ are the same. This means that over any pair of strips the use of Simpson's rule gives a truncation error of

$$\tfrac{1}{90} h^5 f''''(0)$$

If we approximate over $2n$ strips we apply the rule n times; this leads to the size of the principal error term being

$$\text{error} \approx n \times \frac{h^5}{90} \times M$$

where M is the maximum value of $f''''(x)$ in the interval. That is M is a number such that for all x in the interval (a, b)

$$\left| f''''(x) \right| \le M$$

If we are finding the area between the limits a and b then

$$h = \frac{b-a}{2n}$$

or $n = \dfrac{b-a}{2h}$

Hence

$$\begin{aligned} \text{error} &\approx \left(\frac{b-a}{2h} \right) \times \frac{h^5}{90} \times M \\ &= (b-a) \times \frac{h^4}{180} \times M \end{aligned}$$

This means that doubling the number of strips will reduce the error by a factor of 16 when using Simpson's Rule.