Type I and type II errors

The very process of testing a hypothesis indicates that there is a possibility of making an error. The hypothesis test is made at a level of significance. If the significance level is 5% and the alternative hypothesis passes the test, then we can say that there is at least a 5% possibility that the alternative hypothesis is in fact false. The null hypothesis has been rejected in this instance but there is a 5% possibility that it was in fact true.

This illustrates one type of error, but two types are possible:

Type I error:
the error of rejecting the null hypothesis H_0 even though H_0 was true.

Type II error:
the error of accepting the null hypothesis H_0 even though H_0 was false.

Actually, these are conditional probabilities and are written:

P(type I error) = P(reject H_0 | H_0 is true)
P(type II error) = P(accept H_0 | H_0 is false)

The type I error just is the significance level of the test. The way to reduce the possibility of a type I error is to reduce the significance level. The significance level can never be reduced to zero, and the smaller the significance level the greater the possibility of a type II error.

Whilst the type I error, as discussed, just is the significance level of the test, there is the complication is that you may be asked to find the significance level.

Example

A random variable X is normally distributed with mean μ and standard deviation 9. The null hypothesis $\mu = 7$ is to be tested against the alternative hypothesis $\mu < 7$. A random sample of 20 is taken and the null hypothesis is to be rejected if the sample mean is less than or equal to 3.9. Find the probability of a type I error.

Solution

The probability of a type I error just is the significance level of the test which we were being asked to find.

$H_0 : \quad \mu = 7$
$H_1 : \quad \mu < 7$

$n = 16$
$\alpha = ?$

The test is to accept H_0 if $\overline{X} > 3.9$

$P(\text{Type I error}) = P(\text{Reject } H_0 | H_0 \text{ is true})$
$= \text{significance level, } \alpha$

$X \sim N(\mu, 9^2)$
$\overline{X} \sim N\left(\mu, \frac{9^2}{20}\right) = N(\mu, 4.05)$
$z = \frac{|X - \mu|}{\sigma}$
$= \frac{7 - 3.9}{\sqrt{4.05}}$
$= 1.540$

P(z < 1.540) = 0.9337
$P(z < 1.540) = 0.9337$
$\therefore$ P(reject H_0 | H_0 is true) $= 1 - 0.9337 = 0.0663 = 6.6\%(2.S.F.)$

To find a type II error requires a definite alternative hypothesis. It is not possible to assess the conditional probability that the null hypothesis has been accepted whilst false unless we have an alternative hypothesis regarding the population distribution. The following example illustrates this.

Example

Glue is being retailed in 0.5 litre cans. The quality control manager is concerned that too much glue is being put into each can. He takes a random sample of 200 cans with results $\sum X = 105.3$, $\sum X^2 = 59.1$. Test the null hypothesis $\mu = 0.5$ against the alternative hypothesis $\mu > 0.5$ at the 1% significance level. Also calculate the probability of making a type II error when the sample size is 200 given that in fact $\mu = 0.53$. Use the unbiased estimate of the variance derived from the sample.

Solution

The first requirement is a calculation of the unbiased sample variance.

$$\overline{X} = \frac{\sum X}{n} = \frac{105.3}{200} = 0.5265$$

$$S^2 = \frac{\sum X^2}{n} - \overline{X})^2 = \frac{59.1}{200} - (0.5265)^2$$
$$= 0.01829775...$$

$$s^2 = \frac{n}{n-1} S^2$$
$$= \frac{200}{199} \times 0.01829775...$$
$$= 0.018389698...$$
$$= 0.0184(3.S.F.)$$

To test the hypothesis:

$H_0 :\quad \mu = 0.5$
$H_1 :\quad \mu > 0.5$
This is a one-tailed test
The significance level is $\alpha = 0.01$

We take s^2 as an estimate of σ^2

$$X \sim N(0.5, 0183...)$$
$$\therefore\ \overline{X} \sim N\left(0.5, \frac{0.0183...}{200}\right) = N(0.5, 0.000091948...)$$

$$z_{test} = \frac{X-\mu}{\sigma}$$
$$= \frac{0.5265 - 0.5}{\sqrt{0.000091948...}}$$
$$= 2.765(2.D.P.)$$

The critical value for the test is

$$z_{1\%} = 2.328$$

Therefore,

$$z_{test} > z_{1\%}$$

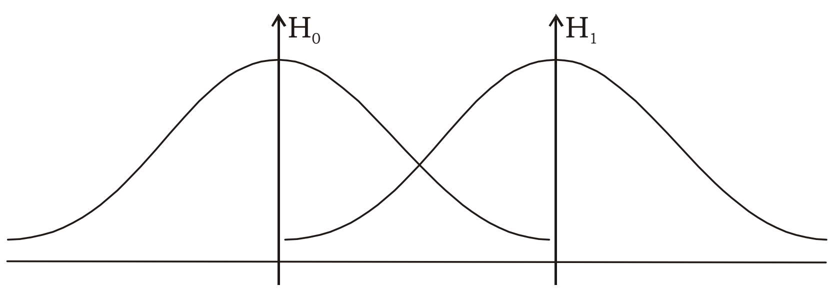
Hence, the test is significant

Reject H_0, accept H_1

For the type II error

$$\begin{aligned} P(\text{Type II}) &= P(\text{Accept } H_0 | H_0 \text{ is false}) \\ &= P(\text{Accept } H_0 | H_1 \text{ is true}) \\ &= P(\text{Accept } \mu = 0.5 | \mu = 0.53) \end{aligned}$$

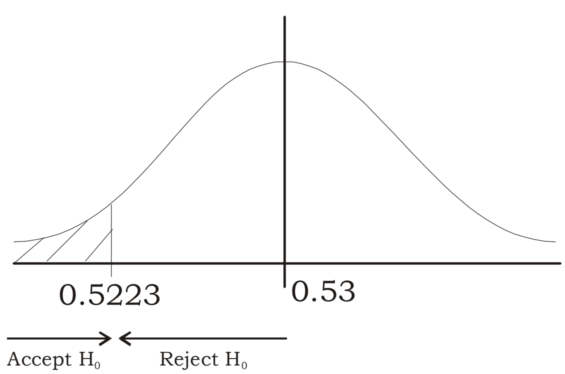
This requires us to compare the distributions for $\mu = 0.5$ and $\mu = 0.53$. The 1% significance level corresponds to $z = 2.328$. At the 1% significance level we accept H_0 if $\bar{X} < 2.328\hat{\sigma}$ where $\hat{\sigma}$ is the estimate of the population standard deviation. Here $\widehat{\sigma} = \hat{\sigma} = \sqrt{\frac{0.0183...}{200}} = 0.00958...$



Hence we accept H_0 up to the value:

$$\begin{aligned} z_{critical} &= 0.5 + 2.328 \, x \, 0.00958... \\ &= 0.5223. \end{aligned}$$

That is:



The probability of a type II error corresponds to the shaded area under the distribution when we assume μ = 0.53.

Under H_1 : μ = 0.53, which has z-value

$$\begin{aligned} z &= \frac{\mu - X}{\sigma} \\ &= \frac{0.53 - 0.5223}{0.009588...} \\ &= 0.803(3.D.P) \end{aligned}$$

Note here that we retain the assumption that $\hat{\sigma}$ = 0.009588...

Hence,

$$\begin{aligned} &P(\text{Type II error}) = P(z < -0.803) \\ &= 1 - P(z < 0.803) \\ &= 1 - 0.7889 \\ &= 0.2111 \\ &= 0.21(2.S.F.) \end{aligned}$$

An estimate of a type I or type II error can involve a Binomial distribution.

Example

A four-sided die is thought to be biased to test this by throwing the die 8 times.

The null hypothesis is p = $\frac{1}{4}$ where p = probability of throwing a four is tested against the alternative hypothesis p > $\frac{1}{4}$. The null hypothesis will be rejected if the number of fours is 4 or more. Calculate to 3 decimal places the probability of making

(i) a type I error
(ii) a type II error if in fact $p = \frac{5}{8}$

Assume $p = \frac{1}{4}$ then $X \sim B\left(8, \frac{1}{4}\right)$

H_0 : $\mu = \frac{1}{4}$

H_0 : $\mu > \frac{1}{4}$

(i) $\quad P(\text{Type I error}) = P(\text{Reject } H_0 | H_0 \text{ is false})$

$$= P(X \geq 4)$$
$$= 1 - P(X \leq 3)$$
$$= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3)$$
$$= 1 - {}^8C_0(3/4)^8 + {}^8C_1(3/4)^7(1/4) + {}^8C_2(3/4)^6(1/4)^2 + {}^8C_3(3/4)^5(1/4)^3$$
$$= 1 - (0.100129 + 0.266958 + 0.311462 + 0.207642)$$
$$= 0.113799$$
$$= 0.114(3.D.P.)$$

(ii) $\quad P(\text{type II error}) = P(\text{accept } H_0 | H_0 \text{ is false})$

$$= P\left(\text{accept } p = 1/4 \middle| p = 5/8\right)$$

Given $p = 5/8$ we have $X \sim B\left(8, 5/8\right)$

We require, $P(X \leq 3)$

$$P(X \leq 3) = C_0(3/8)^8 + {}^8C_1(3/8)^7(5/8) + {}^8C_2(3/8)^6(5/8)^2 + {}^8C_3(3/8)^5(5/8)^3$$
$$= 0.000391 + 0.005214 + 0.030416 + 0.101388$$
$$= 0.137409$$
$$= 0.137(3.D.P.)$$

<u>Example continued...</u>

It is in fact decided that the hypothesis shall be tested by throwing the die 100 times. The null hypothesis will be rejected if the number of fours is greater than k.

Find the value of k required when the probability of making a type I error is 0.005.

$n = 100$

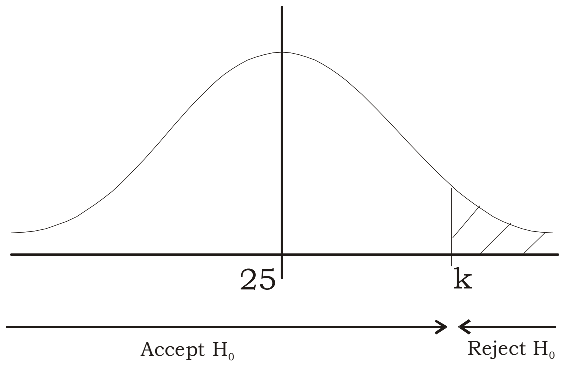
$p = 1/4$

$X \sim B\left(100, 1/4\right)$

But $n > 30$, $np = 25 \geq 5$ and p is sufficiently close to 0.5

So, we approximate by using $X \sim N(25, 18.75)$

X ~ B(100, $1/4$)



Our task is to find the value of *k* that makes the shaded area = 0.005.

The *z*-value corresponding to a probability of 0.005 is $z = 2.575$.

We will ignore the continuity correction at this stage and take account of it later.

Since

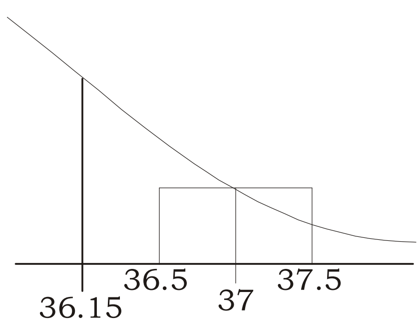
$$z = \frac{X - \mu}{\sigma}$$

We have $2.575 = \dfrac{k - 25}{\sqrt{18.75}}$

Therefore,

$$\begin{aligned} k &= \sqrt{18.75} \times 2.575 + 25 \\ &= 36.15 \end{aligned}$$

k must be a suitable integer greater than 36.15.



Since we are approximating a discrete probability distribution by a continuous one we must consider the possibility that the rectangle corresponding to the next integer, 37, might contain the value 36.15. However, as the diagram illustrates, it does not, so we conclude

$k = 37$