Vector Algebra

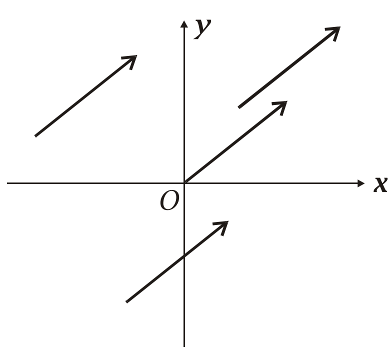
What vectors are

A vector is any object that has both size and direction. Vectors are distinguished from scalars, which are objects with size (i.e. magnitude) only. The distinction between vectors and scalars is of crucial importance. Physics, which describes the interactions of objects in the real world is impossible without the use of vectors. The distinction between speed and velocity is a standard example of the distinction between a scalar and a vector. Imagine a car travelling around a circular track at constant speed. Its speed does not change, but as it is constantly changing its direction so is velocity is constantly changing. Velocity is defined as change of speed or change of direction or both, and change of velocity is acceleration. Velocity and acceleration are examples of vectors. Distance travelled is a scalar, but distance travelled in a certain direction, called displacement, is a vector. Distance travelled is relevant to, for example, the question of fuel consumption, but displacement is relevant to the question – Where am I? Other common scalars are number, mass and volume; other common vectors are force and momentum. Strictly speaking there are two kinds of vector.

(1) There is a **displacement vector**, which is any object specified by only its size and direction, and not by its point of application.

(2) There is also a **position vector**, which is an object specified by its size, direction *and* point of application – that is, its starting point.

We begin by studying mainly displacement vectors. Any vector having the same magnitude and direction is the same displacement vector regardless of where they start. Displacement vectors can be represented diagrammatically by arrows.



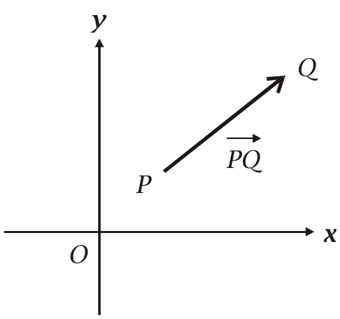
These vectors have the same magnitude and direction, and are consequently the same displacement vector. We ignore the fact that they have different positions.

Representation of vectors

In mathematics we frequently have more than one way of referring to the same mathematical object. We need these different ways because each representation presents the object in a different light. The different representations help us to understand the object in different ways. No one representation captures all that we need to know about the object. In dealing with vectors there are five basic representations that have to be learnt first. There are others, which come at a later stage.

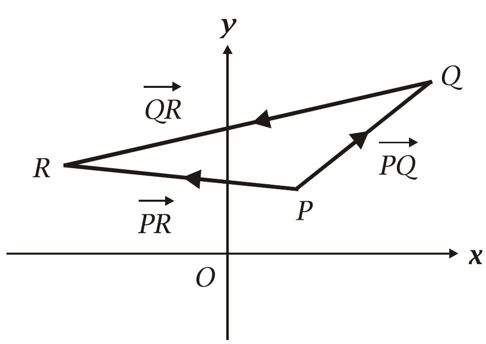
Coordinate representation

The vector $\overrightarrow{PQ}$ is the vector joining the point P (base) to the point Q (tip).



Example (1)

(*a*) The diagram shows three vectors $\overrightarrow{PQ}, \overrightarrow{QR}$ and $\overrightarrow{PR}$. Write an equation expressing $\overrightarrow{PR}$ in terms of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.



(*b*) In what contexts is the coordinate representation of vectors useful?

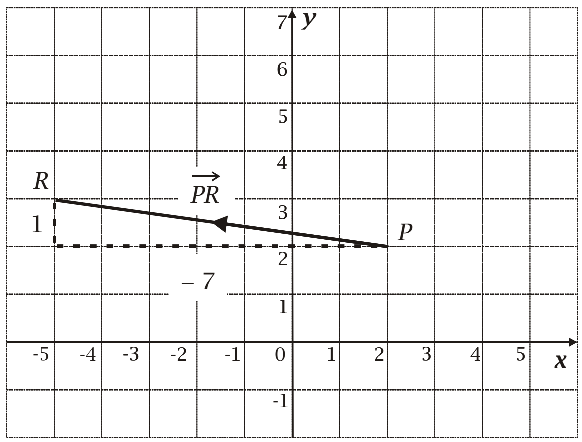
(*c*) If $P = (2,2)$ and $R = (-5,3)$ find the coordinate representation of the vector $\overrightarrow{PR}$.

Solution

(*a*) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$

(*b*) As this example illustrates the coordinate representation of a vector as a directed line (arrow) joining one point to another is useful when we start with a problem in geometry.

(*c*)



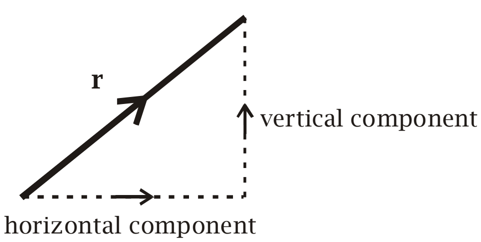
The diagram shows that the displacement vector $\overrightarrow{PR}$ is give by

$\overrightarrow{PR} = (-7,1)$

It is the displacement given by going back 7 in the *x*-direction and up 1 in the *y*-direction.

Component representation of vectors

Vectors can be resolved into horizontal and vertical *components*.

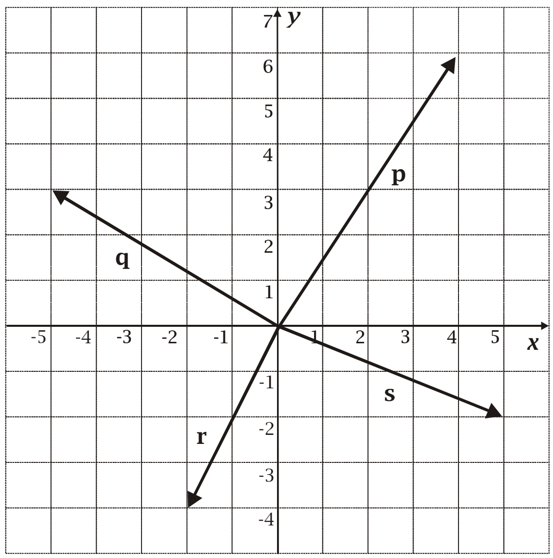


As the above diagram shows, we can think of a vector as the sum of its horizontal and vertical components.

$\mathbf{r}$ = horizontal component of $\mathbf{r}$ + vertical component of $\mathbf{r}$.

Example (2)

Examine the following diagram showing four vectors **p**, **q**, **r** and **s**. Write down the horizontal and vertical components of each of these vectors.



Solution

Vector	Horizontal component	Vertical component
	x	y
p	4	6
q	–5	3
r	–2	–4
s	5	–2

As the table shows we may let x be the horizontal component of a vector and y is its vertical component. We have various ways of representing this information algebraically.

Row vector

(x, y)

This symbol represents the horizontal component x and the vertical component y of a vector as a pair of numbers placed inside a bracket. The order of the pair is fixed – the horizontal component comes first. This is called a *row vector*.

Example (2) continued

Write each of the vectors **p**, **q**, **r** and **s** in example (2) as row vectors

Solution

$\mathbf{p} = (4,6)$

$\mathbf{q} = (-5,3)$

$\mathbf{r} = (-2,-4)$

$\mathbf{s} = (5,-2)$

Column Vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

This is also the vector that is equivalent to a horizontal displacement of x followed by a vertical displacement of y. This symbol has exactly the same meaning as the row vector (x, y) but is in column form. We use both forms of representation because certain operations are easier with column vectors, and certain other operations are easier with row vectors.

Example (2) continued

Write each of the vectors **p**, **q**, **r** and **s** in example (2) as column vectors and show their equivalence to the corresponding row vector.

Solution

$\mathbf{p} = (4,6) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

$\mathbf{q} = (-5,3) = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

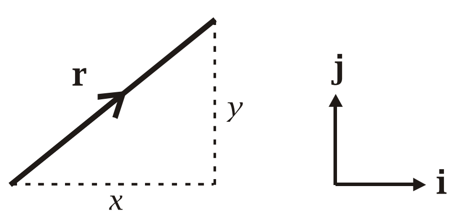
$\mathbf{r} = (-2,-4) = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$

$\mathbf{s} = (5,-2) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

i, j notation

We use the symbols **i** and **j** to represent the x and y directions respectively. So **i** is a unit vector (length 1 unit) in the horizontal direction and **j** is a unit vector in the vertical direction.

$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$



Example (2) continued

Write each of the vectors **p**, **q**, **r** and **s** in example (2) as column vectors and show their equivalence to the corresponding row and column vectors.

$$\mathbf{p} = (4,6) = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = 4\mathbf{i} + 6\mathbf{j}$$

$$\mathbf{q} = (-5,3) = \begin{pmatrix} -5 \\ 3 \end{pmatrix} = -5\mathbf{i} + 3\mathbf{j}$$

$$\mathbf{r} = (-2,-4) = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = -2\mathbf{i} - 4\mathbf{j}$$

$$\mathbf{s} = (5,-2) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} = 5\mathbf{i} - 2\mathbf{j}$$

Example (3)

Write the vectors **i** and **j** as row and column vectors

Solution

$$\mathbf{i} = (1,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathbf{j} = (0,1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The vectors **i** and **j** are called the *basis* vectors. Any two-dimensional vector can be written as a *linear combination* of these two vectors.

Example (4)

Fill in the missing gaps in the following line

$$\mathbf{s} = 5\mathbf{i} - 2\mathbf{j} = \ldots.\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \ldots.\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution

$$\mathbf{s} = 5\mathbf{i} - 2\mathbf{j} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Here we have shown how the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are multiplied by scalars 5 and –2 respectively to give the vector **s**.

Abstract Representation

Throughout these examples we have also implicitly been using another way of representing a vector, simply by a letter printed in **bold**. For example, we used this in the phrase, "the vectors **p**,

q, **r** and **s**". The symbol **r** is used to represent the vector, which is (here) a *two-dimensional* object with a magnitude and direction. This uses the convention that in typed text a vector is written in **bold**. In hand written text a vector is usually underlined.

How vectors appear in text

In textbooks vectors are printed with bold type.

$\mathbf{r}$ = the vector $\mathbf{r} = (x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$

In handwritten text vectors are underlined.

$\underline{r}$ = the vector $\underline{r} = (x, y) = \begin{pmatrix} x \\ y \end{pmatrix} = x\underline{i} + y\underline{j}$

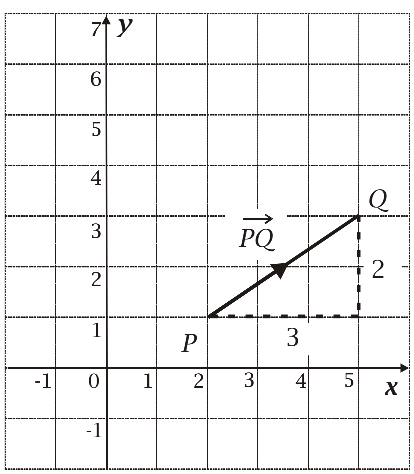
The reason why we underline the vector in hand-written text is because it is very hard to make a difference when we write by hand between a symbol in **bold** and one not. We need to mark a distinction between vectors and scalars precisely because they are different. For example, we cannot add a vector to a scalar, and showing the difference prevents casual errors.

Example (5)

Let the vector $\mathbf{r}$ be the displacement vector from the point $P = (2,1)$ to $Q = (5,3)$.

Write **r** in coordinate form, as a row vector, as a column vector and in **i**, **j** notation. How would this differ if the answer were written by hand?

Solution



$\mathbf{r} = \overrightarrow{PQ} = (3,2) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\mathbf{i} + 2\mathbf{j}$

In hand-written text this would be $\underline{r} = \overrightarrow{PQ} = (3,2) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\underline{i} + 2\underline{j}$.

Addition and subtraction of vectors

When adding and subtracting vectors we employ the rule that we add (or subtract) the horizontal and vertical components separately. Given

$\mathbf{p} = (x_1, y_1)$ and $\mathbf{q} = (x_2, y_2)$

then

$$\mathbf{p} + \mathbf{q} = (x_1 + x_2, y_1 + y_2) = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j}$$

Whether this is done in row, column or **i**, **j** form, we are just adding the components.

Example (6)

Given

$$\begin{aligned} \mathbf{p} &= 2\mathbf{i} - 3\mathbf{j} \\ \mathbf{q} &= -\mathbf{i} + 2\mathbf{j} \\ \mathbf{r} &= -3\mathbf{i} + \mathbf{j} \end{aligned}$$

(*a*) Find **p** + **q** in column form.

(*b*) Find **p** – **r** in row form.

(*c*) Find **q** – **r** in **i**, **j** form.

Solution

(*a*) $\mathbf{p} + \mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(*b*) $\mathbf{p} - \mathbf{r} = (2, -3) - (-3, 1) = (5, -4)$

(*c*) $\mathbf{q} - \mathbf{r} = (-\mathbf{i} + 2\mathbf{j}) - (-3\mathbf{i} + \mathbf{j}) = 2\mathbf{i} + \mathbf{j}$

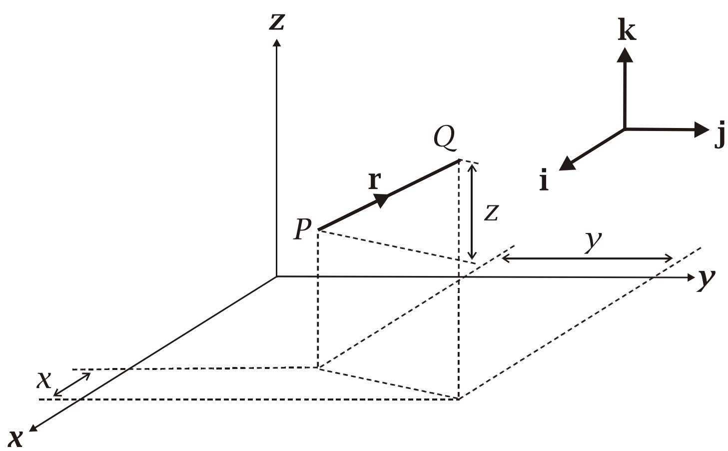
In three dimensions

The discussion has so far been in two dimensions; the extension to three dimensions is as follows. Let $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represent unit vectors in the x-axis, y-axis and z-axis respectively. Then basis vectors for $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are written

$$\mathbf{i} = (1,0,0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{j} = (0,1,0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \mathbf{k} = (0,0.1) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

A vector **r** is written in terms of its components as

$$\mathbf{r} = \overrightarrow{PQ} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



We add and subtract three-dimensional vectors in the same way that we add and subtract them in two dimensions, namely by adding and subtracting their components.

Example (7)

You are given

$\mathbf{p} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$
$\mathbf{q} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$
$\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

(*a*) Find **p** + **q** – **r** in column form.

(*b*) Find **p** + **r** – **q** in row form.

(*c*) Find **q** – **r** – **p** in **i**, **j** form.

Solution

(*a*) $\mathbf{p} + \mathbf{q} - \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix}$

(*b*) $\mathbf{p} + \mathbf{r} - \mathbf{q} = (1,1,-3) + (3,2,-1) - (-1,-3,2) = (5,6,-6)$

(*c*) $\mathbf{q} - \mathbf{r} - \mathbf{p} = (-\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) - (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -5\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$

The Magnitude of a Vector

The magnitude (size or length) of a vector $\mathbf{r}$ is denoted by the use of the modulus symbol by $|\mathbf{r}|$ and is found by using Pythagoras's theorem.

In two dimensions, if $\mathbf{r} = (x, y)$ then $|\mathbf{r}| = \sqrt{x^2 + y^2}$.

In three dimensions, if $\mathbf{r} = (x, y, z)$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$.

Example (8)

Find the size of the following vectors

$\mathbf{p} = 3\mathbf{i} + 4\mathbf{j}$

$\mathbf{q} = 12\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

Solution

$|\mathbf{p}| = \sqrt{3^2 + 4^2} = 5$

$|\mathbf{q}| = \sqrt{12^2 + 3^2 + 4^2} = 13$

Unit vectors

A unit vector is a vector with length = 1. A unit vector in the same direction as the vector $\mathbf{r}$ is denoted by $\hat{\mathbf{r}}$ ("$\mathbf{r}$ hat") and is found by

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}$$

Example (9)

Find a unit vector in the same direction as the vector $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Solution

$|\mathbf{p}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$$\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$