Vector calculus

Scalar Field

For example, let (x, y) denote a position in two-dimensional space and let $P(x, y)$ represent the pressure at that point.

The variable *P* is one-dimensional quantity and is a function of the two variables *x* and *y*. That is

$$P = P(x, y)$$

It is an example of a scalar field. Since it is a function of two variables, it is an example of a two dimensional scalar field.

A three dimensional scalar field would be a function of three variables and an *n* dimensional scalar field is a function of *n* variables.

In general a *n*-dimensional scalar field is a function.

$$\phi \begin{cases} \mathbb{R}^n \rightarrow \mathbb{R} \\ (x_1, x_2, ..x_n) \rightarrow \phi(x_1, x_2 ... x_n) \end{cases}$$

Contour curves

Suppose we have a two-dimensional scalar field $\phi\ (x, y)$. Then a curve will be defined by each specific value that $\phi\ (x, y)$ can take. The curve $\phi\ (x, y) = k$ is called a contour curve of the scalar field.

Note that the contour curves of a temperature field are called isotherms and the contour curves of a pressure field are called isobars.

Example

Let

$\mathbb{R}^2 \to \mathbb{R}$
$\phi\ (x, y) = x^2 + y^2$

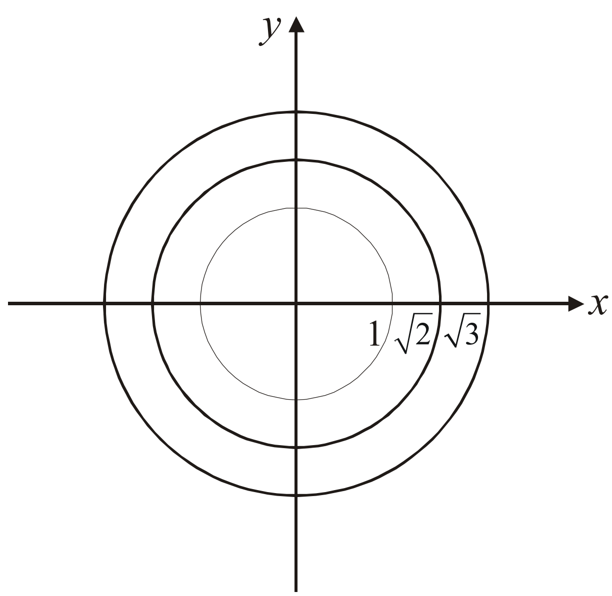
be a scalar field

Sketch the contours given by

$\phi(x, y) = 1$
$\phi(x, y) = 2$
$\phi(x, y) = 3$

Solution

We have $\phi(x, y) = 1$
that is $x^2 + y^2 = 1$
This is the equation of the circle with centre the origin 4 radius1. Similary
$\phi(x, y) = 2$ is a circle with radius $\sqrt{2}$ and $\phi(x, y) = 3$ is a circle with radius $\sqrt{3}$.



A contour map is a map of a surface given by its contours.

If we make a contour map by taking a constant interval between each contour; e.g.

$$\phi(x, y) = 10$$
$$\phi(x, y) = 20$$
$$\phi(x, y) = 30$$

where the difference between successive contours is always 10 units, then the rate of increase or decrease of a scalar field is related to the closeness of the contour curves. The closer the contour curves are together the faster the scalar field is changing.

The idea of a contour curve can be generalised to 3-dimensions.

$$\phi(x, y, z) = k$$

This will give pictorially a series of contour surfaces. For example, if $\phi(x, y, z) = x^2 + y^2 + z^2$ then the contour surfaces given by $x^2 + y^2 + z^2 = k$ for different values of k are a series of rested spheres.

A contour surface can be defined for a *n*-dimensional scalar field

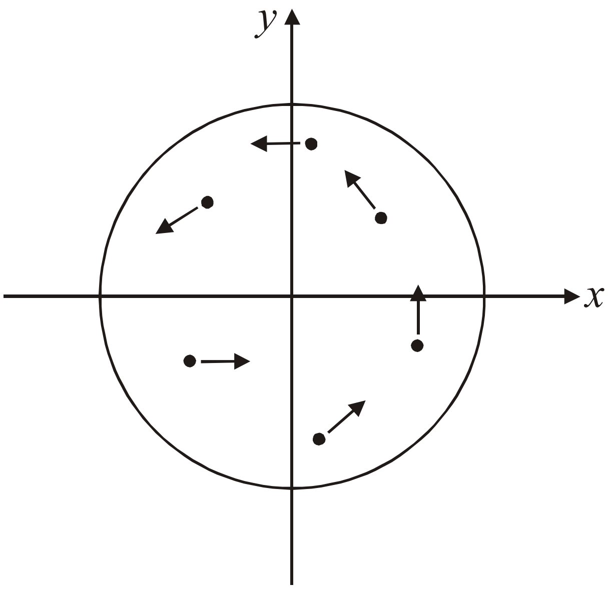
$$\phi(x, x_2 ... x_n)$$

but it is not possible to visualise this in 3-dimensional space.

Vector Field

A vector field is a mapping (function) from one vector space to another.

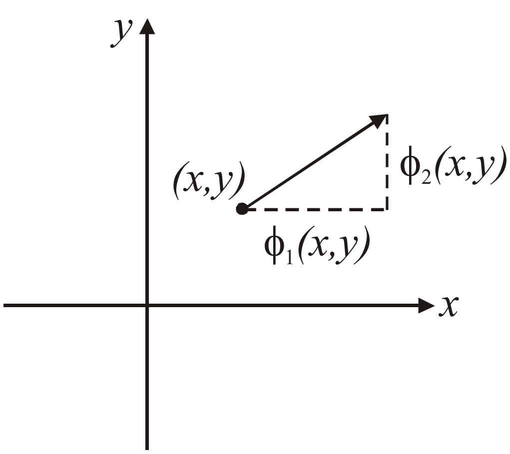
For example, suppose we represent an ocean as a two-dimensional disk and the direction and magnitude of the current on the surface of the ocean by a two dimensional vector, then the function that assigns to each point on the surface of the ocean its current vector is a vector field.



This is the vector field

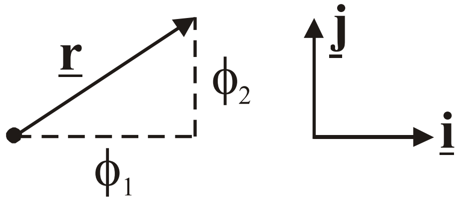
$\mathbb{R}^2 \to \mathbb{R}^2$
$(x, y) \to (\phi_1, (x, y), \phi_2(x, y))$

The functions ϕ_1, and ϕ_2 are the components of the vector field and are themselves two-dimensional scalar fields.



We can represent the rector to which the point (x, y) is mapped by a row or column matrix, or by using $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ notation.

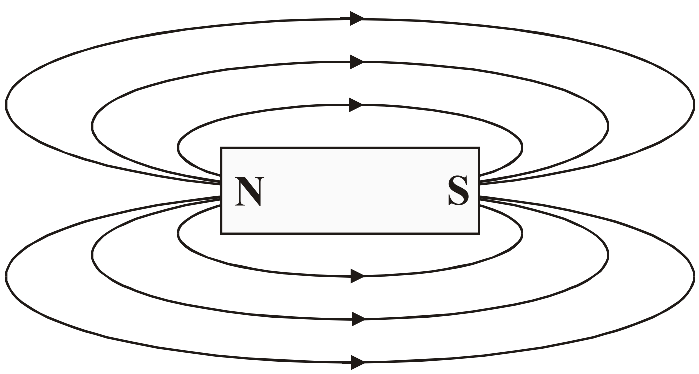
$$\underline{\mathbf{r}} = \left(\phi,(x,y),\phi_2(x,y)\right) = \begin{pmatrix} \phi_1(x,y) \\ \phi_2(x,y) \end{pmatrix} = \phi,(x,y)\,\underline{\mathbf{i}} + \phi_2(x,y)\,\underline{\mathbf{j}}$$



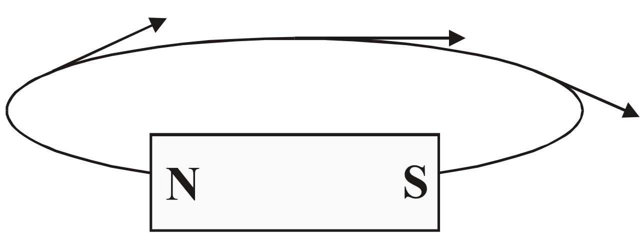
Vector field lines

A vector field line is a continuous curve such that any point on the curve the tangent to the curve is parallel to the direction of the vector field at that point.

For example, vector field lines for a bar magnet could be represented thus: -



The tangent at any point on one of these field lines points in the direction of the vector field there.



Differentiation of scalar and vector products.

Suppose $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ is a vector field

The functions $f_{1,} f_2, f_3$ are its Cartesian coordinates this function. Then the vector field can be differentiated according to the obvious rule.

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_1}{dt}\mathbf{j} + \frac{dt_3}{dt}\mathbf{k}$$

Then differentiation of scalar and vector (cross) products of vectors follows the normal product (Leibniz) rule.

$$\frac{d}{dt}\left(\mathbf{F} \cdot \underline{\mathbf{G}}\right) = \frac{d\mathbf{F}}{dt}\mathbf{G} + \mathbf{F}\frac{d\mathbf{G}}{dt}$$

$$\frac{d}{dt}\left(\mathbf{F} \times \mathbf{G}\right) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

We will prove the result for the cross product. That is

$$\frac{d}{dt}\left(\mathbf{F} \times \mathbf{G}\right) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt} \qquad (*)$$

Let

$\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ *and* $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g\mathbf{k}\pi$

then the left-hand-side of $(*)$ is

$$\text{LHS} = \frac{d}{dt}\left(\underline{\mathbf{F}\times\mathbf{G}}\right)$$

$$= \frac{d}{dt}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3 & g_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$= \frac{d}{dt}\left\{\left(f_2 g_3 - f_3 g_2\right)\mathbf{i} + \left(f_3 g_1 - f_1 g_3\right)\mathbf{j} + \left(f_1 g_2 - f_2 g_1\right)\mathbf{k}\right\}$$

$$= \left(\frac{d}{dt}\left(f_2 g_3\right) - \frac{d}{dt}\left(f_3 g_2\right)\right)\mathbf{i} + \left(\frac{d}{dt}\left(f_3 g_1\right) - \frac{d}{dt}\left(f_1 g_3\right)\right)\mathbf{j} + \left(\frac{d}{dt}\left(f_1 g_2 - f_2 g_1\right)\right)\mathbf{k}$$

$$= \left(f_2' g_3 + f_2 g_3' - f_3' g_2 - f_3' g_2\right)\mathbf{i} + \left(f_3' g_1 + f_3 g_1' - f_1' g_3 - f_1 g_3'\right)\mathbf{j}$$

$$+ \left(f_1' g_2 + f_1 g_2' - f_2' g_1 - f_2 g_1'\right)\mathbf{k}$$

However the right-hand side of (*) is

$$\text{RHS} = \frac{d\mathbf{F}}{dt}\times\mathbf{G} + \mathbf{F}\times\frac{d\mathbf{G}}{dt}$$

$$= \left(f_1'\mathbf{i} + f_2'\mathbf{j} + f_3'\mathbf{k}\right)\times\left(g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}\right) + \left(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}\right)\times\left(g_1'\mathbf{i} + g_2'\mathbf{j} + g_3'\mathbf{k}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \end{vmatrix}$$

$$= \begin{vmatrix} f_2' & f_3' \\ g_2 & g_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3' & f_1' \\ g_3 & g_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1' & f_2' \\ g_1 & g_2 \end{vmatrix}\mathbf{k} + \begin{vmatrix} f_2 & f_3 \\ g_2' & g_3' \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3' & g_1' \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1' & g_2' \end{vmatrix}\mathbf{k}$$

$$= \left(f_2' g_3 - f_3' g_2\right)\mathbf{i} + \left(f_3' g_1 - f_1' g_3\right)\mathbf{j} + \left(f_1' g_2 - f_2 g_1\right)\mathbf{k}$$

$$+ \left(f_2 g_3' - f_3 g_2'\right)\mathbf{i} + \left(f_3 g_1' - f_1 g_3'\right)\mathbf{j} + \left(f_1 g_2' - f_2 g_1'\right)\mathbf{k}$$

$$= \text{LHS}$$

The gradient of a scalar field

Consider a two- dimensional scalar field $\phi = \phi(x, y)$

We will define a vector field called the gradient of the scalar field ϕ by,

$$\text{grad}\phi = \frac{\partial \phi}{\partial x}\underline{\mathbf{i}} + \frac{\partial \phi}{\partial y}\underline{\mathbf{j}} = \begin{pmatrix} \dfrac{\partial \phi}{\partial x} \\ \dfrac{\partial \phi}{\partial y} \end{pmatrix}$$

<u>Example</u>

If $\phi = \ln(x+3y)$ find grad ϕ

Evaluate grad ϕ at $(1,1)$

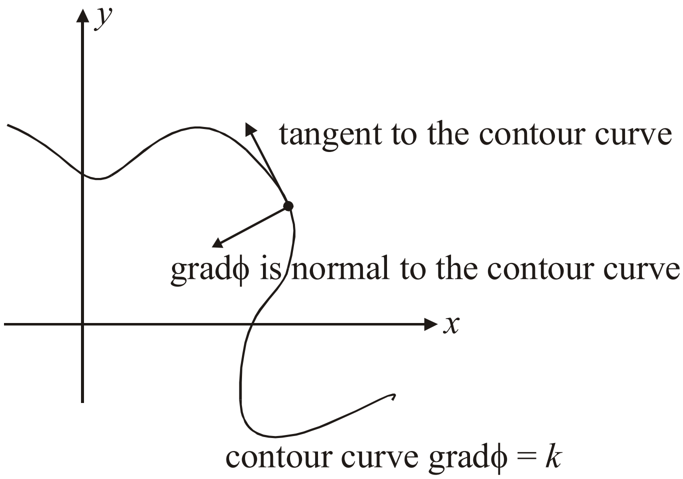
Solution

$$grad\phi = \frac{\partial}{\partial x}\ln(x+3y)\underline{\mathbf{i}} + \frac{\partial}{\partial y}\ln(x+3y)\underline{\mathbf{j}}$$

$$= \left(\frac{1}{x+3y}\right)\underline{\mathbf{i}} + \left(\frac{3}{x+3y}\right)\underline{\mathbf{j}}$$

$$grad\phi\Big|_{(1,1)} = \frac{1}{4}\underline{\mathbf{i}} + \frac{3}{4}\underline{\mathbf{j}}$$

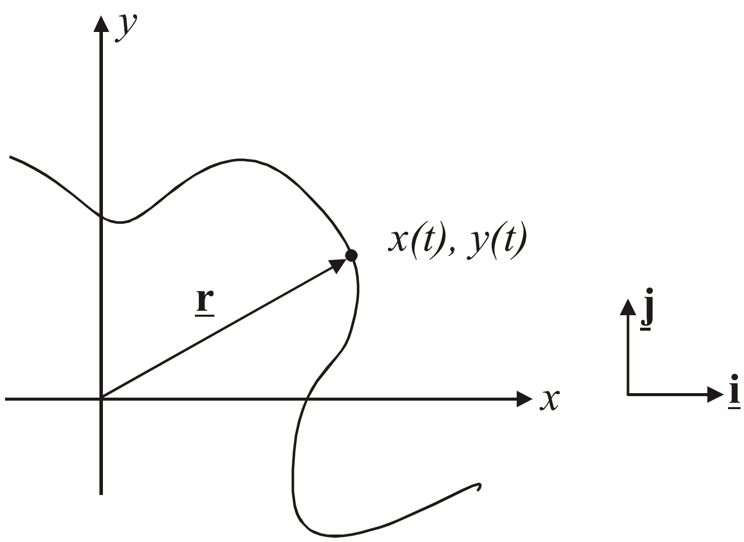
We will now show that the direction of gradϕ at a point is perpendicular to contour curve passing through that point – that is, it points in the direction of the normal to that contour.



Proof

We have $\phi(x, y) = k$

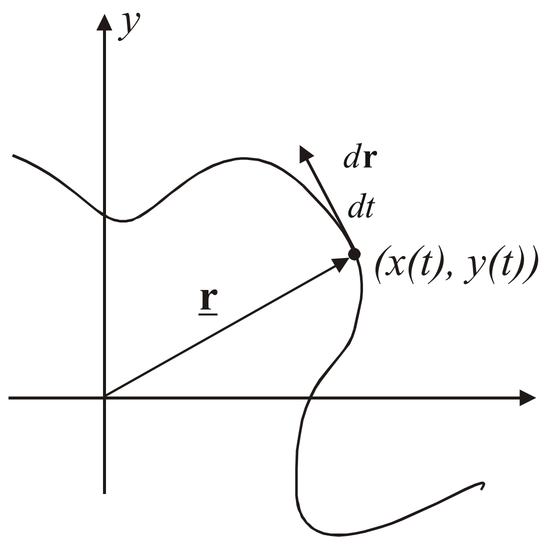
as the equation of a contour curve.

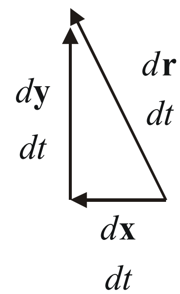


Let $\underline{\mathbf{r}} = (x(t), y(t))$ be a parameterisation of this contour curve.

Then a vector tangent to this contour curve will be

$$\frac{\partial \underline{\mathbf{r}}}{\partial t} = \frac{\partial x}{\partial t}\underline{\mathbf{i}} + \frac{\partial y}{\partial t}\underline{\mathbf{j}}$$





Along this curve $\phi(x,y)=\phi(x(t),y(t))=$ where k is a constant.

Hence, differentiating with respect to t,

$$\frac{\partial\phi}{\partial t}=0$$

However, since ϕ is a function of x and y and these are regarded as functions of t, we can apply the chain rule to differentiate ϕ.

$$\frac{\partial\phi}{\partial t}=\partial\left(\phi(x),\partial(y)\right)$$
$$=\frac{\partial\phi(x)}{\partial t}\underline{\mathbf{i}}+\frac{\partial\phi(y)}{\partial t}\underline{\mathbf{j}}$$
$$=\left(\frac{\partial\phi}{\partial x}\bullet\frac{\partial x}{\partial t}\right)\underline{\mathbf{i}}+\left(\frac{\partial\phi}{\partial y}\bullet\frac{\partial y}{\partial t}\right)\underline{\mathbf{j}}$$

But here $\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial x}$ is the partial denvative of ϕ with respect to x, and likewise $\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial y}$ is the partial denvative of ϕ with respect to y.

So, $\frac{\partial\phi}{\partial t}=\left(\frac{\partial\phi}{\partial x}\bullet\frac{\partial x}{\partial t}\right)\underline{\mathbf{i}}+\left(\frac{\partial\phi}{\partial y}\bullet\frac{\partial y}{\partial t}\right)\underline{\mathbf{j}}$

Since $\frac{\partial \phi}{\partial t} = 0$, this means $\left(\frac{\partial \phi}{\partial x} \bullet \frac{\partial x}{\partial t}\right)\underline{\mathbf{i}} + \left(\frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t}\right)\underline{\mathbf{j}} = 0$

Now the expression $\left(\frac{\partial \phi}{\partial x} \bullet \frac{\partial x}{\partial t}\right)\underline{\mathbf{i}} + \left(\frac{\partial \phi}{\partial y} \bullet \frac{\partial y}{\partial t}\right)\underline{\mathbf{j}}$ is the scalar (dot) product of the two

vectors $\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right), \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$.

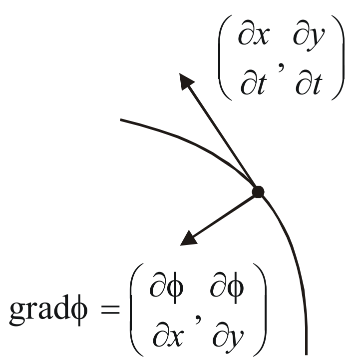
That is $\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) \bullet \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial y}\right) = \left(\frac{\partial \phi}{\partial x} \bullet \frac{\partial x}{\partial t}, \frac{\partial \phi}{\partial y} \bullet \frac{\partial y}{\partial t}\right)$

Hence the dot product of these two vectors is zero.

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) \bullet \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right) = 0$$

Hence the vector $\left(\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial y}\right)$ is perpendicular to the vector

Since the vector $\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}\right)$ is tangent to the contour curve, the vector $grad\,\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$ is normal to it



In three-dimensional space a scalar field is represented by the field function $\phi(x, y, z)$. Its gradient is

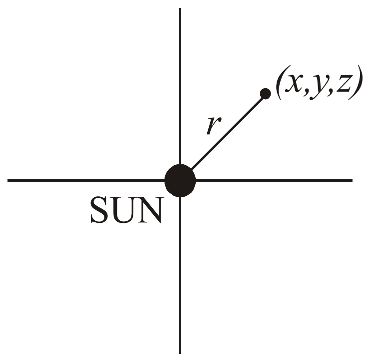
$$grad\phi = \frac{\partial \phi}{\partial x}\underline{\mathbf{i}} + \frac{\partial \phi}{dy}\underline{\mathbf{j}} + \frac{\partial \phi}{\partial z}\underline{\mathbf{k}}$$

Example

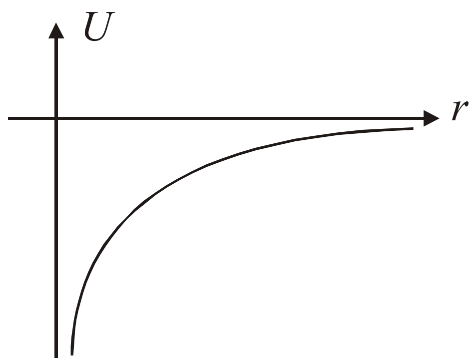
The gravitational potential at a point (x, y, z) of a gravitational field is given by

$$U(x, y, z) = \frac{c}{\sqrt{x^2 + y^2 = z^2}}$$

where c is a constant. For example, the gravitational field surrounding the sun



The gravitational potential is inversely proportional to the distance of the point from the centre of the sun



Find $gradU$ and show that this points in the direction of the centre of the gravitational field.

$$U=\frac{C}{\sqrt{x^2+y^2+z^2}}=C\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial U}{\partial x}=\frac{C}{2}(x^2+y^2+z^2)^{-3/2}2\times=\frac{-\times}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial U}{\partial y}=\frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\frac{\partial U}{\partial z}=\frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$\therefore GradU=\left(\frac{-x}{\left(x^2+y^2+z^2\right)^{3/2}},\frac{-y}{\left(x^2+y^2+z^2\right)^{3/2}},\frac{-z}{\left(x^2+y^2+z^2\right)^{3/2}}\right)$$

$$=-\left(x,y,z\right).\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}$$

So Grad U points in the direction $-\left(x,y,z\right)$.

That is the direction $-\underline{i}-\underline{j}-\underline{k}$.

Hence it points towards the centre of the gravitational field.

We will now show that the magnitude of grade ϕ at the point $\left(x,y,z\right)$ gives the magnitude of the maximum rate of change of ϕ at that point.

[Proof in this text is murky]

The Vector Operator ∇

Instead of grad ϕ we use the expression $\nabla\phi$. The symbol ∇ is pronounced 'del' or 'nabla'.

It stands for

$$\underline{i}\;\frac{\partial}{\partial \text{x}}+$$

And if signifies the operation of finding the first - order partial derivatives of ϕ and the formation of the vector field

$$\nabla\phi = \underline{i}\ \frac{\partial\phi}{\partial x} + \underline{j}\frac{\partial\phi}{\partial y} + \underline{k}\frac{\partial\phi}{\partial z}$$

∇ is not itself a vector, but by applying ∇ to a scalar field ϕ a vector field is defined; hence it is called a differential vector operator.