Vector differential calculus

Scalar Field

For example, let (x, y) denote a position in two-dimensional space and let $P(x, y)$ represent the pressure at that point.

The variable P is one-dimensional quantity and is a function of the two variables x and y. That is

$$P = P(x, y)$$

It is an example of a scalar field. Since it is a function of two variables, it is an example of a two dimensional scalar field.

A three dimensional scalar field would be a function of three variables and an n dimensional scalar field is a function of n variables. In general a n-dimensional scalar field is a function.

$$\phi \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ (x_1, x_2, ..x_n) \to \phi(x_1, x_2 ... x_n) \end{cases}$$

Practical examples of scalar fields in physics are the assignment of a temperature to points of a body, and the pressure of the air of the Earth's atmosphere. These are both mappings from a three dimensional space to a single real number, and so are examples of three dimensional scalar fields.

<u>Example</u>

The Euclidean distance from a point

$$\mathbf{r} = (x, y, z)$$

from a fixed point

$$\mathbf{p} = (x_0, y_0, z_0)$$

is an example of a scalar field. It is given by the formula

$$f(\mathbf{r}) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

Contour Curves

Suppose we have a two-dimensional scalar field $\phi\ (x, y)$. Then a curve will be defined by each specific value that $\phi\ (x, y)$ can take. The curve $\phi\ (x, y) = k$ is called a contour curve of the scalar field.

(Note that the contour curves of a temperature field are called isotherms and the contour curves of a pressure field are called isobars.)

<u>Example</u>

Let

$\mathbb{R}^2 \rightarrow \mathbb{R}$

$\phi\ (x, y) = x^2 + y^2$

be a scalar field

Sketch the contours given by

$\phi(x, y) = 1$

$\phi(x, y) = 2$

$\phi(x, y) = 3$

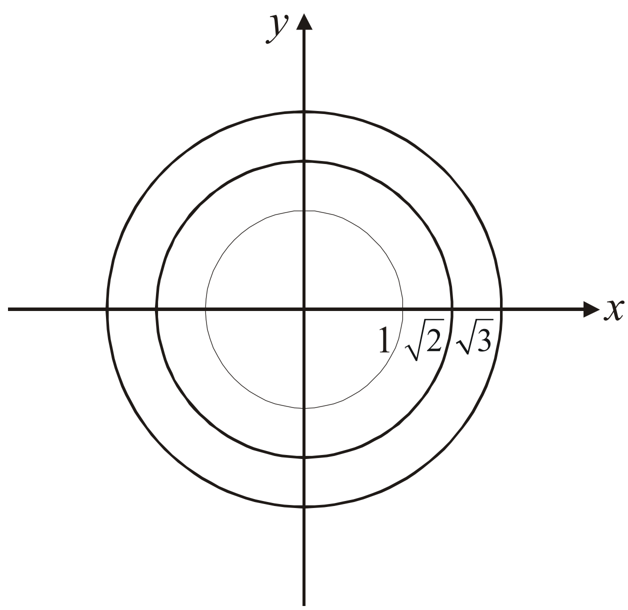
Solution

We have $\phi(x, y) = 1$

that is $x^2 + y^2 = 1$

This is the equation of the circle with centre the origin 4 radius1. Similary

$\phi(x, y) = 2$ is a circle with radius $\sqrt{2}$ and $\phi(x, y) = 3$ is a circle with radius $\sqrt{3}$.



A contour map is a map of a surface given by its contours.

If we make a contour map by taking a constant interval between each contour; e.g.

$\phi(x, y) = 10$
$\phi(x, y) = 20$
$\phi(x, y) = 30$

where the difference between successive contours is always 10 units, then the rate of increase or decrease of a scalar field is related to the closeness of the contour curves. The closer the contour curves are together the faster the scalar field is changing.

Contour Surfaces

The idea of a contour curve can be generalised to 3-dimensions.

$\phi(x, y, z) = k$

This will give pictorially a series of contour surfaces. For example, if $\phi(x, y, z) = x^2 + y^2 + z^2$ then the contour surfaces given by $x^2 + y^2 + z^2 = k$ for different values of k are a series of rested spheres

.A contour surface can be defined for a *n*-dimensional scalar field

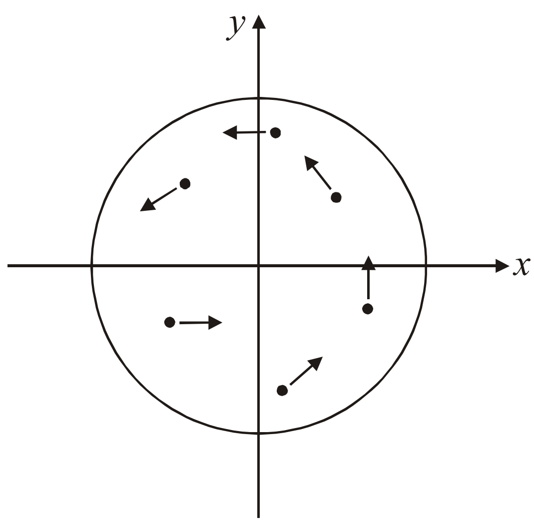
$\phi(x, x_2 \ldots x_n)$

but it is not possible to visualise this in 3-dimensional space.

Vector Field

A vector field is a mapping (function) from one vector space to another.

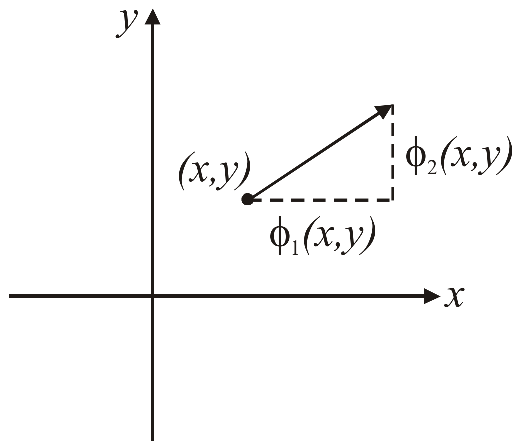
For example, suppose we represent an ocean as a two-dimensional disk and the direction and magnitude of the current on the surface of the ocean by a two dimensional vector, then the function that assigns to each point on the surface of the ocean its current vector is a vector field.



This is the vector field

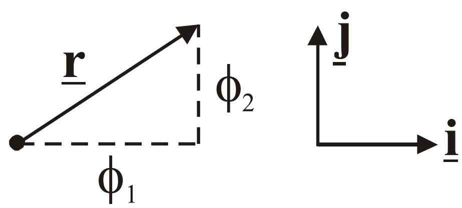
$$\mathbb{R}^2 \to \mathbb{R}^2$$
$$(x, y) \to (\phi_1, (x, y), \phi_2(x, y))$$

The functions ϕ_1, and ϕ_2 are the components of the vector field and are themselves two-dimensional scalar fields.



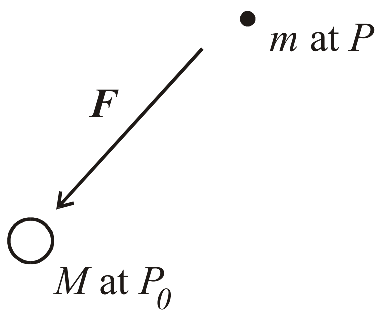
We can represent the rector to which the point (x, y) is mapped by a row or column matrix, or by using $\underline{\mathbf{i}}, \underline{\mathbf{j}}$ notation.

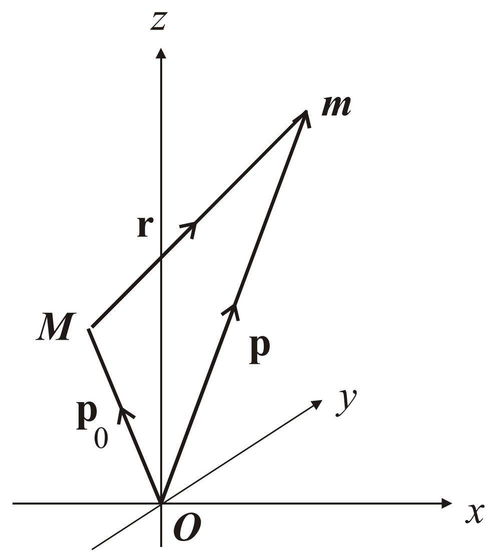
$$\underline{\mathbf{r}} = \left(\phi_1(x,y), \phi_2(x,y)\right) = \begin{pmatrix} \phi_1(x,y) \\ \phi_2(x,y) \end{pmatrix} = \phi_1(x,y)\underline{\mathbf{i}} + \phi_2(x,y)\underline{\mathbf{j}}$$



Example

Newton's law of gravitation gives rise to a vector field describing the gravitational force acting on a particle within a gravitational field.
Suppose a particle of mass M is fixed at a point P_0 and that another smaller particle of mass m is situated at a point P in space.





Let F be the force acting on m by virtue of the gravitational field created by M. By Newton's law of gravitation, the magnitude of this force is

$$|\mathbf{F}| = \frac{GMm}{r^2}$$

where G is the gravitational constant $\left(G = 6.67 \times 10^{-11}\, Nm^2 kg^{-3}\right)$ and r is the distance between the two particles. The force is directed from P to P_0. Hence, this defines a vector field. The distance, r, is given by

$$r = |\mathbf{p} - \mathbf{p}_0| = |(x, y, z) - (x_0, y_0, z_0)| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

The vector from P to P_0 is

$$\mathbf{r} = \mathbf{p} - \mathbf{p_0} = (x - x_0, y - y_0, z - z_0) = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}$$

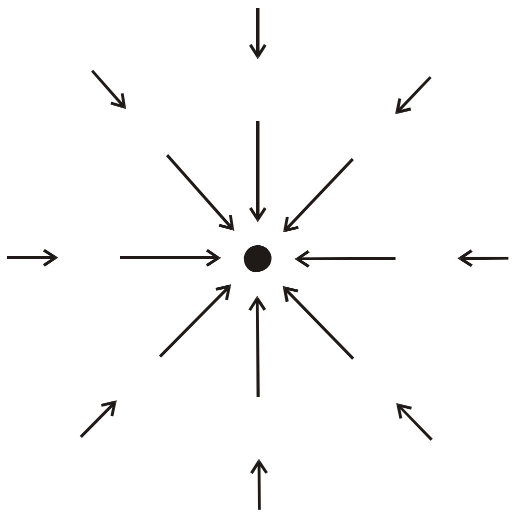
The direction of the force is in the opposite direction to this vector, so a unit vector in this direction is given by

$$\hat{\mathbf{r}} = -\frac{1}{r}\mathbf{r}$$

The force is

$$\mathbf{F} = f(x, y, z) = \frac{GMm}{r^2} \times -\frac{1}{r}\mathbf{r} = -\frac{GMm}{r^3}\left[(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}\right]$$

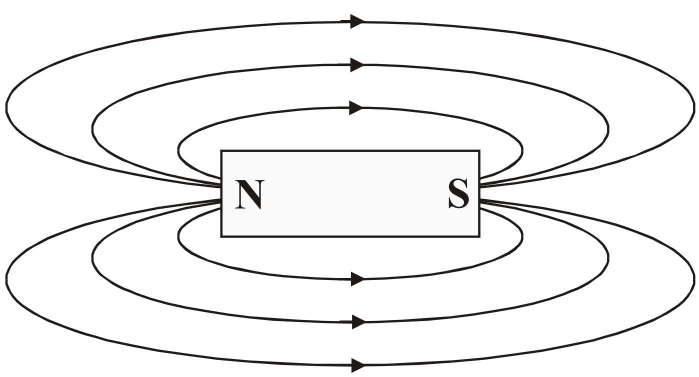
Since the position *P* of the particle *m* can vary, this defines a vector field.



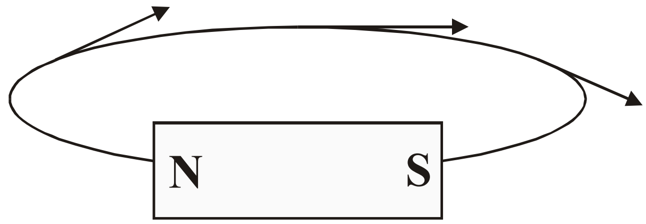
Vector field lines

A vector field line is a continuous curve such that any point on the curve the tangent to the curve is parallel to the direction of the vector field at that point.

For example, vector field lines for a bar magnet could be represented thus: -



The tangent at any point on one of these field lines points in the direction of the vector field there.



Vector functions

We may imagine a particle moving along a contour curve; for example, along the contour curve

$$x^2 + y^2 = 1$$

of the scalar field $\phi\ (x, y) = x^2 + y^2$. This movement may be a function of time, or some other parameter. More general the position of a particle in space may be given by a vector function

$$\mathbf{v}(t) = \left(v_1(t), v_2(t), v_3(t)\right)$$

where *t* is a parameter. This is a vector function of 3 dimensions, but clearly the concept could be applied to 2 dimensions or spaces of dimension greater than 3. Such a vector function may not be always continuous or differentiable, but if it is, then its derivative will be defined in the usual way, by

$$\frac{d}{dt}\mathbf{v}(t) = \left(\frac{d}{dt}v_1(t), \frac{d}{dt}v_2(t), \frac{d}{dt}v_3(t)\right) \qquad \mathbf{v}'(t) = \left(v_1'(t), v_2'(t), v_3'(t)\right)$$

That is, by differentiating each of the components of the vector function.

The usual rules for differentiating scalar multiples and sum of functions applies to vector functions. Specifically

$$\frac{d}{dt}(c\mathbf{v}) = c\frac{d}{dt}\mathbf{v} \qquad (c\mathbf{v})' = c\mathbf{v}'$$

$$\frac{d}{dt}(\mathbf{u} + \mathbf{v}) = \frac{d}{dt}\mathbf{u} + \frac{d}{dt}\mathbf{v} \qquad (\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}'$$

Differentiation of scalar and vector products.

Suppose $\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ is a vector field

The functions $f_{1,}f_2, f_3$ are its Cartesian coordinates this function. Then the vector field can be differentiated according to the obvious rule.

$$\frac{d\mathbf{F}}{dt} = \frac{df_1}{dt}\mathbf{i} + \frac{df_1}{dt}\mathbf{j} + \frac{dt_3}{dt}\mathbf{k}$$

Then differentiation of scalar and vector (cross) products of vectors follows the normal product (Leibniz) rule.

$$\frac{d}{dt}(\mathbf{F} \cdot \mathbf{G}) = \frac{d\mathbf{F}}{dt}\mathbf{G} + \mathbf{F}\frac{d\mathbf{G}}{dt}$$

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

We will prove the result for the cross product. That is

$$\frac{d}{dt}(\mathbf{F} \times \mathbf{G}) = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt} \qquad (*)$$

Let

$\mathbf{F} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ *and* $\mathbf{G} = g_1\mathbf{i} + g_2\mathbf{j} + g_2\mathbf{k}$

then the left-hand-side of $(*)$ is

$$\text{LHS} = \frac{d}{dt}(\underline{\mathbf{F} \times \mathbf{G}})$$
$$= \frac{d}{dt}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \begin{vmatrix} f_2 & f_3 \\ g_2 & g_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3 & g_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix}$$

$$= \frac{d}{dt}\left\{ \left(f_2 g_3 - f_3 g_2\right)\mathbf{i} + \left(f_3 g_1 - f_1 g_3\right)\mathbf{j} + \left(f_1 g_2 - f_2 g_1\right)\mathbf{k} \right\}$$

$$= \left(\frac{d}{dt}\left(f_2 g_3\right) - \frac{d}{dt}\left(f_3 g_2\right) \right)\mathbf{i} + \left(\frac{d}{dt}\left(f_3 g_1\right) - \frac{d}{dt}\left(f_1 g_3\right) \right)\mathbf{j} + \left(\frac{d}{dt}\left(f_1 g_2 - f_2 g_1\right) \right)\mathbf{k}$$

$$= \left(f_2' g_3 + f_2 g_3' - f_3' g_2 - f_3' g_2 \right)\mathbf{i} + \left(f_3' g_1 + f_3 g_1' - f_1' g_3 - f_1 g_3' \right)\mathbf{j}$$

$$+ \left(f_1' g_2 + f_1 g_2' - f_2' g_1 - f_2 g_1' \right)\mathbf{k}$$

However the right-hand side of (*) is

$$\text{RHS} = \frac{d\mathbf{F}}{dt} \times \mathbf{G} + \mathbf{F} \times \frac{d\mathbf{G}}{dt}$$

$$= \left(f_1'\mathbf{i} + f_2'\mathbf{j} + f_3'\mathbf{k} \right) \times \left(g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k} \right) + \left(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \right) \times \left(g_1'\mathbf{i} + g_2'\mathbf{j} + g_3'\mathbf{k} \right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1' & f_2' & f_3' \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_1 & f_2 & f_3 \\ g_1' & g_2' & g_3' \end{vmatrix}$$

$$= \begin{vmatrix} f_2' & f_3' \\ g_2 & g_3 \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3' & f_1' \\ g_3 & g_1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1' & f_2' \\ g_1 & g_2 \end{vmatrix}\mathbf{k} + \begin{vmatrix} f_2 & f_3 \\ g_2' & g_3' \end{vmatrix}\mathbf{i} + \begin{vmatrix} f_3 & f_1 \\ g_3' & g_1' \end{vmatrix}\mathbf{j} + \begin{vmatrix} f_1 & f_2 \\ g_1' & g_2' \end{vmatrix}\mathbf{k}$$

$$= \left(f_2' g_3 - f_3' g_2 \right)\mathbf{i} + \left(f_3' g_1 - f_1' g_3 \right)\mathbf{j} + \left(f_1' g_2 - f_2 g_1 \right)\mathbf{k}$$

$$+ \left(f_2 g_3' - f_3 g_2' \right)\mathbf{i} + \left(f_3 g_1' - f_1 g_3' \right)\mathbf{j} + \left(f_1 g_2' - f_2 g_1' \right)\mathbf{k}$$

$$= \text{LHS}$$