The Vector Equation of the Straight Line

Prerequisites

You should understand what a displacement vector is and that displacement vectors can be added and subtracted by adding and subtracting their components.

Example (1)

Given

$\mathbf{r} = 2\mathbf{i} - 4\mathbf{k}$ $\qquad$ $\mathbf{s} = -\mathbf{j} + 2\mathbf{k}$

Find $\mathbf{r} - 2\mathbf{s}$

Solution

$\mathbf{r} - 2\mathbf{s} = 2\mathbf{i} - 4\mathbf{k} - 2(-\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}$

Vector equations

We can rearrange equations containing vectors in the same way that we can rearrange regular equations.

Example (2)

Make **a** the subject of the equation $6\mathbf{a} - 3\mathbf{b} + \frac{1}{2}\mathbf{c} = \mathbf{d}$

Solution

$6\mathbf{a} = \mathbf{d} + 3\mathbf{b} - \frac{1}{2}\mathbf{c}$

$\mathbf{a} = \frac{1}{6}\left(\mathbf{d} + 3\mathbf{b} - \frac{1}{2}\mathbf{c}\right)$

This means that equations involving vectors can be solved by the usual algebraic manipulations.

Example (3)

Solve the vector equation $\mathbf{p} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Solution

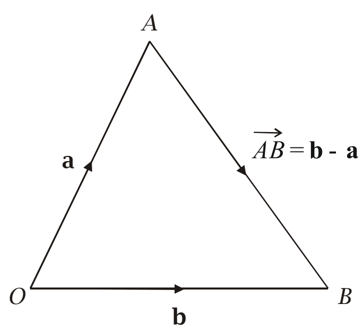
$$\mathbf{p} + \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Subtracting $\begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$ from both sides of this equation

$$\mathbf{p} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-3-6 \\ 2+0-1 \\ -3+3+1 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}$$

Vector equation of the straight line

Vectors are introduced so that we can describe objects that exist in two and three-dimensional space. One such object is a straight line. We can use vector algebra to find the vector equation of the straight line joining two points A and B. Suppose $\mathbf{a} = \overrightarrow{OA}$ is the vector joining the origin O to the point A and $\mathbf{b} = \overrightarrow{OB}$ the vector joining O to B.



Then the vector $\overrightarrow{AB}$ that joins A to B is such that $\mathbf{a} + \overrightarrow{AB} = \mathbf{b}$. Rearranging this equation to solve for $\overrightarrow{AB}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

Thus $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ is the vector from A to B. A straight line is fixed in space by reference to the origin O. So in this equation the vectors **a** and **b** must have fixed points of application, which is the origin. This means that they are *position vectors.*

Example (4)

The position vectors of the points A and B are given by

$\mathbf{a} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k} \qquad \mathbf{b} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

(*a*) Find the vector $\overrightarrow{AB}$ from *A* to *B*.

(*b*) Find the vector $\overrightarrow{BA}$.

(*c*) Find the vector $\mathbf{p} = \mathbf{b} + 2\left(\overrightarrow{BA}\right)$.

(*d*) Make a sketch showing the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$ and $\mathbf{p} = \mathbf{b} + 2\left(\overrightarrow{BA}\right)$, and determine for each whether it is a displacement or position vector.

(*e*) Describe in terms of geometry the vector $\mathbf{r} = \mathbf{b} + s\left(\overrightarrow{BA}\right)$ where *s* is any real number.

Solution

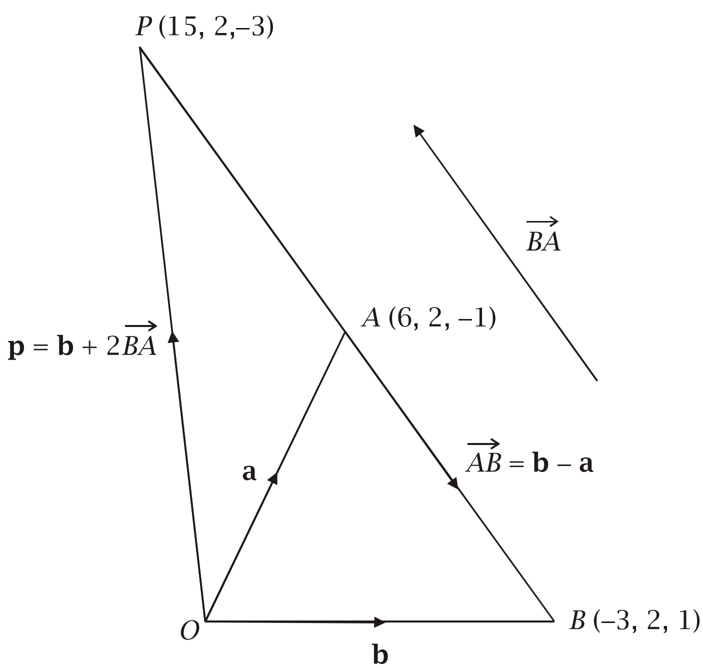
(*a*) It is easier to add vectors in column form.

$$\mathbf{a} = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \qquad \overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ 2 \end{pmatrix} = -9\mathbf{i} + 2\mathbf{k}$$

(*b*) $\overrightarrow{BA} = -\overrightarrow{AB} = -\left(-9\mathbf{i} + 2\mathbf{k}\right) = 9\mathbf{i} - 2\mathbf{k}$

(*c*) $$\mathbf{p} = \mathbf{b} + 2\overrightarrow{BA} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 9 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 15 \\ 2 \\ -3 \end{pmatrix} = 15\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

(*d*) For the sketch we make no attempt to represent any of these vectors in three dimensions – that is we *do not* draw three axes and plot the points *A*, *B* or *P* represented by $\mathbf{b} + 2\left(\overrightarrow{BA}\right)$.



(This diagram shows the *plane* that is determined by the points *O*, *A* and *B*. The point *P* also lies in this plane.)

We are given that $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ are position vectors, meaning that they fix the position of the points *A* and *B* relative to the origin *O*. Likewise the vector

$$\mathbf{p} = \mathbf{b} + 2\left(\overrightarrow{BA}\right)$$

must also be a position vector. It fixes the position of the point *P* given by $\mathbf{p} = 15\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ relative to the origin. But the vector $\overrightarrow{BA}$ is a *displacement* vector. It is the displacement from *B* to *A* and is not fixed relative to *B*. For example we could write

$$\mathbf{q} = \mathbf{a} + 2\left(\overrightarrow{BA}\right)$$

If $\overrightarrow{BA}$ were fixed at *B* this would not make any sense because we would then not be able to interpret adding the vector fixed at *B* to a vector at the point *A* fixed by $\mathbf{a} = \overrightarrow{OA}$. The vector $\overrightarrow{BA}$ here represents any vector that has the same direction and magnitude as $\overrightarrow{BA}$. Likewise $\overrightarrow{AB}$ is a displacement vector. The equation

$$\mathbf{p} = \mathbf{b} + 2\left(\overrightarrow{BA}\right)$$

says $\mathbf{p} = \overrightarrow{OP}$ is the position vector of the point *P* found by adding the position vector of the point *B* given by $\mathbf{b} = \overrightarrow{OB}$ to twice the displacement vector $\overrightarrow{BA}$ that is equal to the displacement from *B* to *A*. However, we remark that it is not necessary at this level to distinguish systematically between position and displacement vectors as the vector algebra simply "works".

(*e*) The equation

$$\mathbf{r} = \mathbf{b} + s\left(\overrightarrow{BA}\right)$$

fixes the position of a point **r** on the line joining *B* to *A*. As *s* varies so this position varies. Therefore, it is the vector equation of the straight line joining *B* to *A*. Since *s* can take negative values, by substituting a negative value of *s* we get the vector equation of the line joining *A* to *B*. Geometrically, these are the same line, but travelled in opposite directions. So the vector equation of the straight line provides information (*a*) about the position of the line in space – i.e. it is the line joining *A* to *B* or, what is the same thing, the line joining *B* to *A*; and (*b*) the optional information about the direction in which a particle might be travelling along that line. Thus the equation

$$\mathbf{r} = \mathbf{b} + t\left(\overrightarrow{AB}\right)$$

for positive values of the parameter *t* shows a particle travelling along a line in the direction from *A* to *B*. The equation

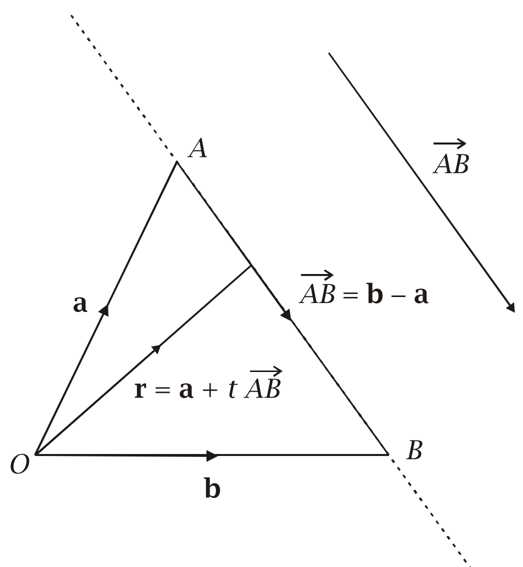
$\mathbf{r} = \mathbf{b} + s\left(\overrightarrow{BA}\right)$

where s is a positive parameter shows a particle travelling alone the *same* line but in the opposite direction from B to A. By using different parameters t or s we have also the possibility that the particles are travelling at a different speed. All of this may be of particular relevance if you are also studying mechanics, which a natural application of vector algebra.

The last example shows that the vector equation of the straight line passing through the points A and B with position vectors $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$ respectively is given by

$\mathbf{r} = \mathbf{b} + t\left(\overrightarrow{AB}\right)$

where t is a real valued parameter.



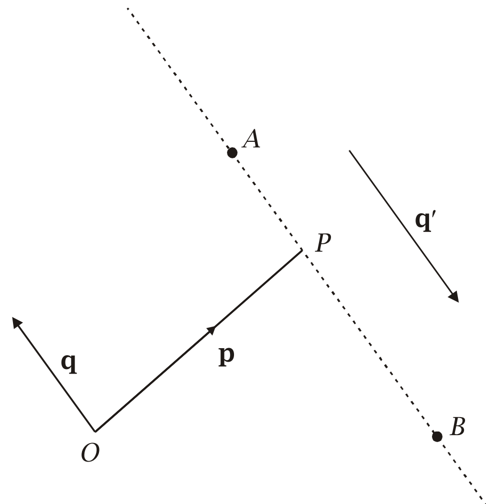
The *same line* is described by any of the following

$\mathbf{r} = \mathbf{b} + s\left(\overrightarrow{BA}\right)$ $\quad$ $\mathbf{r} = \mathbf{a} + t\left(\overrightarrow{AB}\right)$ $\quad$ $\mathbf{r} = \mathbf{a} + s\left(\overrightarrow{BA}\right)$

Thus the most general form of the vector equation of a straight line is

$\mathbf{r} = \mathbf{p} + t\mathbf{q}$

where $\mathbf{p}$ is the position vector of *any* point on the line, $\mathbf{q}$ is a displacement vector of any size having the same direction, forwards or backwards, as the line, and t is a real-valued parameter.



A line is determined by (1) fixing any point P lying on the line and by (2) any displacement vector having the same direction, forwards or backwards, as the line.

Example (5)

The position vectors of the points A and B are given by

$$\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$$

(a) Find the vector equation of the line AB.

(b) Find the position vector of the point C lying on AB such that $|AC| = 2|BC|$.

Solution

(a) $$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

The vector equation of the line AB is

$$\mathbf{r} = \mathbf{a} + t\overrightarrow{AB}$$

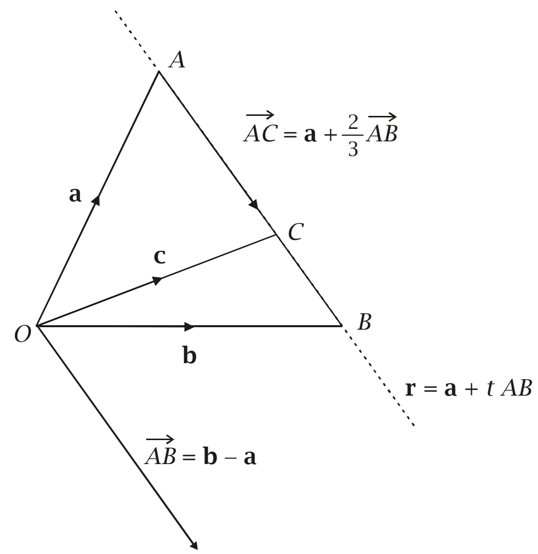
$$\mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

Note, the equation

$$\mathbf{r} = \mathbf{b} + t\overrightarrow{AB} \qquad \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

describes the same line.

(*b*) Since $|AC| = 2|BC|$ then $|AC| = \frac{2}{3}|AB|$



Hence

$$\overrightarrow{AC} = \mathbf{a} + \frac{2}{3}\overrightarrow{AB}$$

$$= \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \frac{14}{3}\mathbf{i} - \frac{13}{3}\mathbf{j} + 7\mathbf{k}$$

Points of Intersection

In two dimensions any two lines that are not parallel, and not collinear, must intersect at a unique point. Hence, given two vector equations of lines in two dimensions, it is possible to find the position vector of their point of intersection.

Example (6)

Let l_1 be given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \lambda \text{ is a real number}$$

and l_2 be given by

$$\mathbf{s} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mu \text{ is a real number}$$

Find their point of intersection.

Solution

Equating the two lines:

$$\begin{pmatrix}2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}4\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\1\end{pmatrix}$$

$2+\lambda=4+2\mu$

$3-\lambda=-1+\mu$

On adding these two equations:

$$5=3+3\mu \qquad \Rightarrow \qquad \mu=\frac{2}{3} \text{ and } \lambda=\frac{10}{3}$$

Substituting $\mu=\frac{2}{3}$ into l_2 or $\lambda=\frac{10}{3}$ into l_1, both give

$$\mathbf{p}=\begin{pmatrix}4\\-1\end{pmatrix}+\frac{2}{3}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}5\frac{1}{3}\\-\frac{1}{3}\end{pmatrix}$$

In two dimensions, equating the equations of two lines gives two equations in two unknowns. Provided one equation is not a multiple of another, this gives a unique solution. In three dimensions there is no necessity that two lines intersect. If two lines do intersect, then equating them gives three equations in two unknowns that do have a unique and consistent solution. If the two lines do not intersect, solving the system of three equations in pairs gives different values for the point of intersection and inconsistent values for the parameters describing the point on the line.

Example (7)

Show that

l_1 given by

$$\mathbf{r}=\begin{pmatrix}3\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\2\end{pmatrix}$$

and l_2 given by

$$\mathbf{r}=\begin{pmatrix}0\\2\\1\end{pmatrix}+s\begin{pmatrix}3\\0\\3\end{pmatrix}$$

do not intersect.

Solution

Suppose they do intersect, then

$$\begin{pmatrix}3\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}0\\2\\1\end{pmatrix}+s\begin{pmatrix}3\\0\\3\end{pmatrix}$$

$3+2t=3s$ (1)

$-1+t=2$ (2)

$1+2t=1+3s$ (3)

From (2) $t=3$

Substituting in (1)

$3+6=3s$

$s=3$

Substituting in (3)

$1+6=1+3s$

$s=2$

The values $s=2$ and $s=3$

The values $s=2$ and $s=3$ are inconsistent, hence there cannot be a point of intersection.

This is a proof by contradiction. In this method of proof, you assume something to be true and deduce from it a contradiction. When you arrive at a contradiction, you conclude that the thing you assumed must be false. Here we prove that there cannot be a point of intersection by, firstly, assuming that there is a point of intersection, and then showing that this leads to a contradiction.

Example (8)

The position vectors of the points A and B are given by

$\mathbf{a}=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$, $\mathbf{b}=-5\mathbf{i}+4\mathbf{j}-\mathbf{k}$

(*a*) Find the vector equation of the line *AB*.

(*b*) The vector equation of the line *L* is

$\mathbf{r}=7\mathbf{i}+p\mathbf{j}-4\mathbf{k}+\mu(3\mathbf{i}+5\mathbf{j}+5\mathbf{k})$

where p is a constant. Given that *AB* and *L* intersect, find p and the point of intersection.

Solution

(*a*) $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} -5 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix}$

The vector equation of the line *AB* is

$$\mathbf{s} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k} + \lambda(-7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

(*b*) The two lines intersect. Therefore

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ p \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$$

$2 - 7\lambda = 7 + 3\mu$ (1)

$1 + 3\lambda = p + 5\mu$ (2)

$3 - 4\lambda = -4 + 5\mu$ (3)

We first solve equations (1) and (3) simultaneously to find λ and μ.

$3\mu + 7\lambda = -5$ $(1)'$

$5\mu + 4\lambda = 7$ $(3)'$

$15\mu + 35\lambda = -25$

$15\mu + 12\lambda = 21$

$23\lambda = -46$

$\lambda = -2 \quad \mu = 3$

Then from equation (2)

$1 - 6 = p + 15$

$p = -20$

The point of intersection is found by substituting either $\lambda = -2$ or $\mu = 3$ into

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ -20 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$$

This gives

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + -2 \begin{pmatrix} -7 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ -20 \\ -4 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix} = 16\mathbf{i} - 5\mathbf{j} + 11\mathbf{k}$$

Parametric and Cartesian equations of the straight line

We have seen that the general form of the vector equation of the straight line is

$\mathbf{r} = \mathbf{a} + t\mathbf{p}$

where **a** is the position vector of *any* point on the line, **p** is a displacement vector of any size having the same direction, forwards or backwards, as the line, and *t* is a real-valued parameter. Suppose that we have

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

$\mathbf{p} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$

Here, *t, x, y, z* are variables: *a, b, c, p, q, r* are constants. Then the vector form can be written

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Uncoupling this equation – that is, separating the three components – gives

$x = a + tp$
$y = b + tq$
$z = c + tr$

This is called the parametric form of the equation of the straight line. The parameter is *t*, and the equations show how the coordinates vary as *t* varies. If we solve these equations for *t* we obtain

$$t = \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$$

This is called the Cartesian equation of the straight line.

Example (9)

Determine the point of intersection between the following pair of lines given in Cartesian form

$$\frac{x+4}{3} = \frac{y-1}{3} = \frac{z+3}{4} \text{ and } \frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$$

Solution

Putting these into parametric form

$$\frac{x+4}{3} = \frac{y-1}{3} = \frac{z+3}{4} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

$$\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

At intersection

$$\begin{pmatrix}-4\\1\\-3\end{pmatrix}+t\begin{pmatrix}3\\3\\4\end{pmatrix}=\begin{pmatrix}-1\\4\\5\end{pmatrix}+s\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$-4+3t=-1+s \qquad (1)$$

$$1+3t=4+s \qquad (2)$$

$$-3+4t=5+2s \qquad (3)$$

$(1)-(2)$ $\quad -5=-5$

$(2)\times 2$ $\quad 2+6t=8+2s \qquad (4)$

$(4)-(3)$ $\quad 5+2t=3$

$$2t=-2$$

$$t=-1$$

Sub in (2) $\quad s=-6$

The point of intersection is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-4\\1\\-3\end{pmatrix}-\begin{pmatrix}3\\3\\4\end{pmatrix}=\begin{pmatrix}-7\\-2\\-7\end{pmatrix}$$