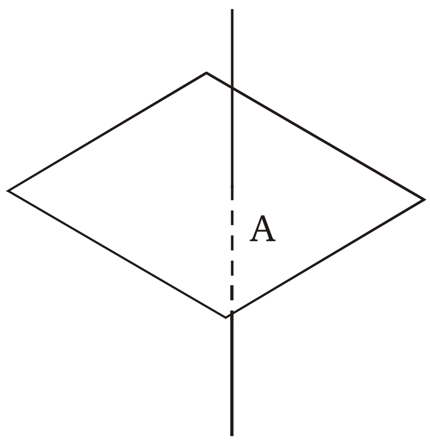
Vector planes

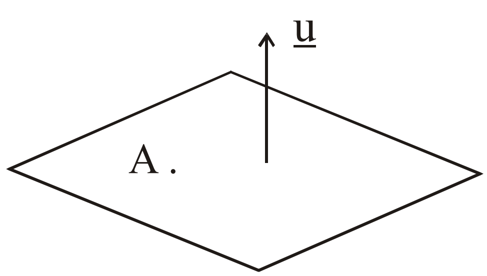
The theory of vector planes

A plane can be specified by:

(1) a point lying in the plane
(2) a line that is perpendicular to the plane.



It is not necessary that the line passes through the given point A, and it is sufficient to know the direction of the line – that is, any vector**, $\underline{\mathbf{u}}$,** that is parallel to the line.



This suggests that a plane can be given a vectorial form. This is indeed the case. The general equation of a plane in vector form is

$$\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$$

Where $\underline{\mathbf{r}}$ is the position vector of any point lying in the plane and $\underline{\mathbf{u}}$ is any vector that is perpendicular to the plane; k is a real number.

To prove this, let A be any point in a plane π with position vector $\underline{r}$. Let $\underline{\mathbf{u}}$ be a vector perpendicular to π.

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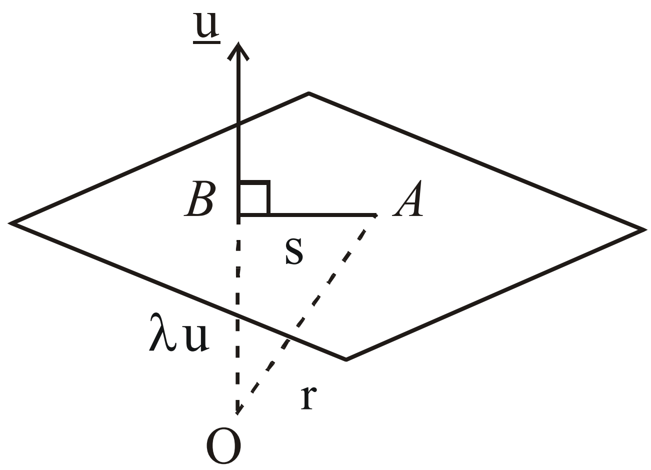

Let B be a point lying in the plane π such that the position vector of B is $\lambda\underline{\mathbf{u}}$.

Let $\underline{\mathbf{s}}$ be the vector joining B to A. Then

$\lambda\underline{\mathbf{u}} + \underline{\mathbf{s}} = \underline{\mathbf{r}}$
$\therefore \underline{\mathbf{s}} = \underline{\mathbf{r}} - \lambda\underline{\mathbf{u}}$

Since $\underline{\mathbf{s}}$ lies in the plane π $\underline{\mathbf{s}}$ is the perpendicular to $\underline{\mathbf{u}}$; hence

$\underline{\mathbf{s}} \cdot \underline{\mathbf{u}} = 0$
$\therefore (\underline{\mathbf{r}} - \lambda\underline{\mathbf{u}}) \bullet \underline{\mathbf{u}} = 0$
$\therefore \underline{\mathbf{r}} \cdot \underline{\mathbf{u}} - \lambda\underline{\mathbf{u}} \cdot \underline{\mathbf{u}} = 0$

Since λ is a real number and $\underline{\mathbf{u}} \cdot \underline{\mathbf{u}} = |\underline{\mathbf{u}}|^2$ which is also a real number $\lambda\underline{\mathbf{u}} \cdot \underline{\mathbf{u}} = k$ where k is a real number

Hence
$\therefore \underline{\mathbf{r}} \cdot \underline{\mathbf{u}} - k = 0$
and
$\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$ is the vector equation of a plane.

Suppose $k = 0$, then this implies that $\underline{\mathbf{r}}$ and $\underline{\mathbf{u}}$ are perpedicular.
This is only possible if the vector plane passes through the origin.

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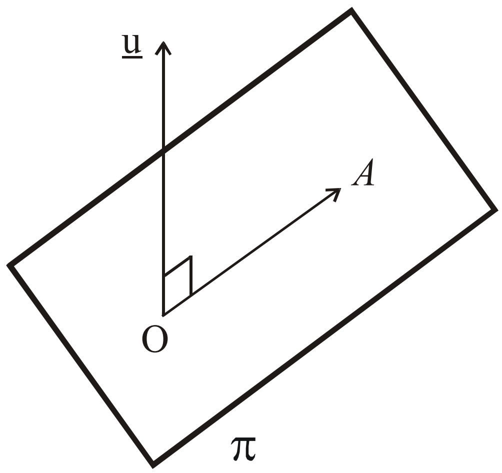

The position vector, $\underline{\mathbf{r}}$, of any point, A, that lies in the plane π also lies in the plane and is consequently perpendicular to the vector $\underline{\mathbf{u}}$ that is perpendicular to the plane.

Distance of a plane from the origin

So given $\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$ then **u** determines the perpendicular normal to the plane – it may be thought of as directed from the origin. k determines how far away from the origin this plane is. It is a distance along the normal vector – a scaling of it.

$\mathbf{r} \cdot \mathbf{u} = k$

Let $\mathbf{u} = (x, y, z)$

Then $\mathbf{r} = \lambda(x, y, z)$

$\lambda(x, y, z) \cdot (x, y, z) = k$

$$\lambda = \frac{k}{x^2 + y^2 + z^2}$$

Let d be the distance from the origin of the plane, then

$|\mathbf{u}| = \sqrt{x^2 + y^2 + z^2}$

$$d = \frac{k}{x^2 + y^2 + z^2}|\mathbf{u}| = \frac{k}{\sqrt{x^2 + y^2 + z^2}} = \frac{k}{|\mathbf{u}|}$$

Cartesian form

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Vector planes also have a Cartesian form. That is, a plane in 3 dimensions is an object defined by a linear relationship between the Cartesian coordinates, x, y, z.

It is an easy matter to interchange between the Cartesian and vector plane form of a plane in 3 dimensions.

The vector equations of a plane is:-

$\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$

Let $\underline{\mathbf{r}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the position vector of any point A lying in the plane.

Let $\underline{\mathbf{u}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

be any vector that is perpendicular to the plane. Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = k$$

$$\therefore \ ax + by + cz = k$$

which gives the Cartesian form of the plane. It is equally easily to exchange from the Cartesian equation to the vector plane. Simply reverse the process.

Whilst this embodies the entire theory of vector planes these simple definitions give rise to a large number of problems.

(1) To determine whether a line lies in plane, is parallel to a plane, or intersects a plane, and to find the point of intersection of a line and a plane when it exists.

If a line lies in a plane, then every point on the line satisfies the vector equation of the plane.

Thus if $\underline{\mathbf{s}} = \underline{\mathbf{a}} + t\underline{\mathbf{b}}$ is the equation of the line, and

$\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$

is the vector equation of a plane,

then $\underline{\mathbf{s}}$ lies in the plane if, and only if, $\underline{\mathbf{s}} \cdot \underline{\mathbf{u}} = k$

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Example

Let $\underline{\mathbf{r}} \cdot (3\underline{\mathbf{i}} + \underline{\mathbf{j}} - 2\underline{\mathbf{k}}) = 5$ be the equation of a plane. Show that

$$\underline{\mathbf{s}} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

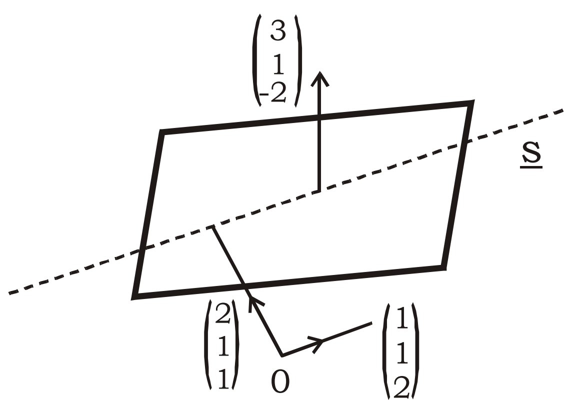
lies in the plane.

Answer

substituting $\underline{\mathbf{s}}$ for $\underline{\mathbf{r}}$ in $\underline{\mathbf{r}} \cdot (3\underline{\mathbf{i}} + \underline{\mathbf{j}} - 2\underline{\mathbf{k}})$ gives

$$\left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$$

Hence $\underline{\mathbf{s}}$ lies in the plane.



Looking at the vector equation of a plane:

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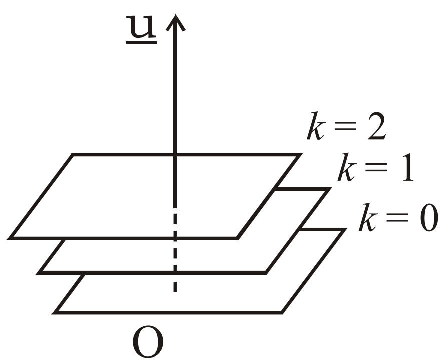
If

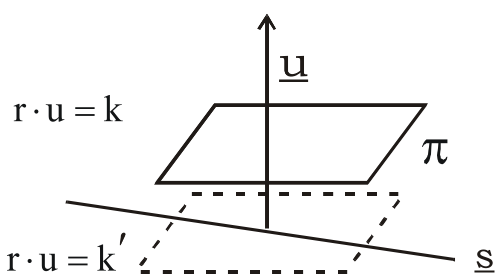
$\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} \;= k$

where k is real, then for a given vector $\underline{\mathbf{u}}$, substituting different values of k defines a family of planes, π_k, each parallel to the other.

This is because each plane is perpendicular to $\underline{\mathbf{u}}$.



Suppose $\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} \;=\; k$ defines a vector plane, π, then, if a line $\underline{\mathbf{s}} = \underline{\mathbf{a}} + \; t\underline{\mathbf{b}}$ is parallel to this plane but does not lie in it, then $\underline{\mathbf{s}} \cdot \underline{\mathbf{u}} = k'$ where $k' \;\neq\; k$.



If a line does not lie in a plane and is not parallel to it either then, in 3 dimensions it must intersect with it.

The unique point of intersection can be found by substituting the equation of the line into the equation of the plane and solving for the parameter.

It is best to clarify this by an example.

Example

Find the point of intersection of the line

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$$\underline{\mathbf{s}} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

with the plane $\underline{\mathbf{r}} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

Answer

Substituting $\underline{s}$ for $\underline{r}$ in the vector equation of the plane gives:-

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \right\} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$$

$$\therefore \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$$

$$\therefore \ (6+1-4) + \lambda\ (3+3-4) = 5$$

$$3 + 2\lambda = 5$$

$$2\lambda = 2$$

$$\lambda = 1$$

$$\therefore \ \underline{\mathbf{s}} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 4 \end{pmatrix} = 3\underline{\mathbf{i}} + 4\underline{\mathbf{j}} + 4\underline{\mathbf{k}}$$

is the point of intersection of the line with the plane.

(2) To find the line of intersection of two non-parallel planes.

The following diagram illustrates this problem

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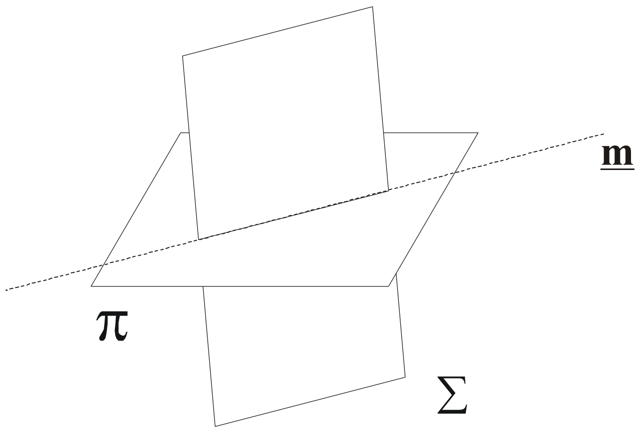
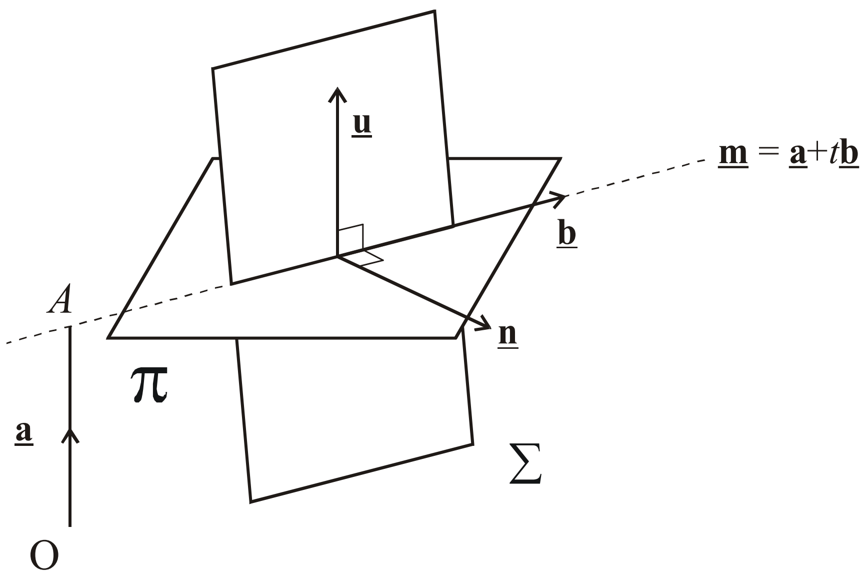


The non-parallel planes, π and Σ, intersect in a unique line $\underline{\mathbf{m}}$.

To find this line, let π be defined by $\underline{\mathbf{r}} \cdot \underline{\mathbf{u}} = k$ and Σ by $\underline{\mathbf{s}} \cdot \underline{\mathbf{n}} = k'$.
Then $\underline{\mathbf{m}}$ satisfies both vector plane equations.
Let $\underline{\mathbf{m}} = \underline{\mathbf{a}} + t\underline{\mathbf{b}}$.
The vector $\underline{\mathbf{b}}$ is parallel to the vector $\underline{\mathbf{m}}$.



The vector $\underline{\mathbf{u}}$ is perpendicular to π, and the vector $\underline{\mathbf{n}}$ is perpendicular to Σ. Therefore, $\underline{\mathbf{b}}$ is perdendicular to both $\underline{\mathbf{u}}$ and $\underline{\mathbf{n}}$. Hence, a suitable choice of $\underline{\mathbf{b}}$ is the cross product of $\underline{\mathbf{u}}$ and $\underline{\mathbf{n}}$.

$\underline{\mathbf{b}} = \underline{\mathbf{u}} \times \underline{\mathbf{n}}$

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The vector $\underline{\mathbf{a}}$ defines the position vector of a point A lying in both planes.

To find $\underline{\mathbf{a}}$ we must use this fact, by substituting $\underline{\mathbf{a}} = \begin{pmatrix} e \\ f \\ g \end{pmatrix}$ into the equations for both vector planes and solve simultaneously.

<u>Example</u>

Find the equation of the line of intersection of the vector plane.

π $\quad\quad \underline{\mathbf{r}} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

and Σ $\quad\quad \underline{\mathbf{r}} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 2$

Answer

Let the line be given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + t\underline{\mathbf{b}}$ then

$\underline{\mathbf{b}} = (3,1,-2) \times (1,-2,2)$

$$= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 3 & 1 & -2 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \underline{\mathbf{i}} \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} + \underline{\mathbf{j}} \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} + \underline{\mathbf{k}} \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= \underline{\mathbf{i}}(2-4) + \mathbf{j}(-2-6) + \underline{\mathbf{k}}(-6-1) = -2\underline{\mathbf{i}} - 8\underline{\mathbf{j}} - 7\underline{\mathbf{k}} = \begin{pmatrix} -2 \\ -8 \\ -7 \end{pmatrix}$$

Let $\underline{\mathbf{a}} = \begin{pmatrix} e \\ f \\ g \end{pmatrix}$

Then on substituting in the equations for π and $\sum$

$$\begin{pmatrix} e \\ f \\ g \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5 \text{ and } \begin{pmatrix} e \\ f \\ g \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 2$$

$3e + f - 2g = 5$ and $e - 2f + 2g = 2$

let e = 2 then $\underline{a} = \begin{pmatrix} 2 & \\ 4 \times 2 & -7 \\ \frac{7\times 2}{2} & -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

on adding

$4e - f = 7$

$f = 4e - 7$

then

$3e + (4e - 7) - 2g = 5$

$7e - 2g = 12$

$2g = 7e - 12$

$g = \frac{7e}{2} - 6$

hence $\underline{\mathbf{r}} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -8 \\ -7 \end{pmatrix}$

is the equation of the line of intersection of the plane π and $\sum$.

We can check this by substituting $\underline{\mathbf{r}}$ back into the equations for the two planes and showing that $\underline{\mathbf{r}}$ satisfies both. Thus

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -8 \\ -7 \end{pmatrix} \right\} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -8 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

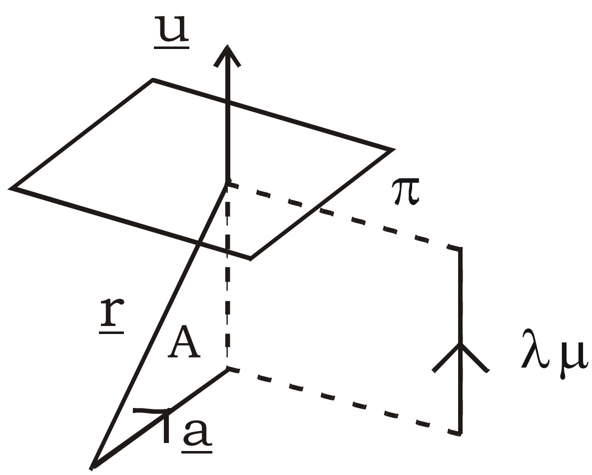
$$= 6 + 1 - 2 + \lambda(-6 - 8 + 14)$$
$$= 5$$

Hence $\underline{\mathbf{r}}$ lies in the plane π. Likewise

$$\left\{\begin{pmatrix}2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}-2\\-8\\-7\end{pmatrix}\right\}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\-8\\-7\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix} = 2-2+\lambda(-2+16-14)$$

Hence $\underline{\mathbf{r}}$ lies in the plane Σ.

(3) To find the perpendicular distance from a point to a plane



The perpendicular length will be lenght of a vector that is perpendicular to the plane.

Therefore, to find this length let the perpendicular vector be a scalar multiple of the vector $\underline{\mathbf{u}}$ that is found in the definition of of π.

That is, suppose π is given by $\underline{\mathbf{r}}\cdot\underline{\mathbf{u}} = k$ and the position vector of the point A is $\underline{\mathbf{a}}$. Then the vector

$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{u}}$

lies in the plane π. Hence by solving

$(\underline{\mathbf{a}}+\lambda\underline{\mathbf{u}})\bullet\underline{\mathbf{u}} = k$

we find λ. Then the perpendicular distance from A to π is

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$\lambda|\underline{\mathbf{u}}|$

That is $\lambda \times$ the length of the vector $\underline{\mathbf{u}}$

Example

Find the perpendicular distance of the point $(-1,1,-3)$ from the plane $\underline{\mathbf{r}} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

Answer

Let $\underline{\mathbf{r}} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

then $\underline{\mathbf{r}} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

that is $\left\{ \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \right\} \cdot \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

$\therefore \quad \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 5$

$-3 + 1 + 6 + \lambda(9 + 1 + 4) = 5$

$4 + 14\lambda = 5$

$14\lambda = 1$

$\lambda = \frac{1}{14}$

$|\underline{\mathbf{u}}| = |(3,1,-2)|$

$= \sqrt{9+1+4} = \sqrt{14}$

$\therefore$ perpendicular distance $= \lambda|\underline{\mathbf{u}}| = \frac{1}{14} \times \sqrt{14} = \sqrt{\frac{1}{14}}$

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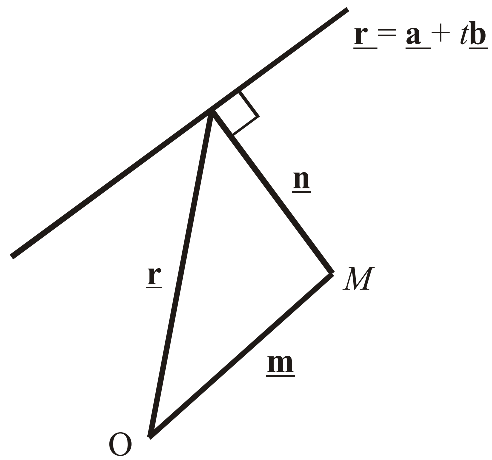
(4) To find the perpendicular distance from a point to a line

To find the perpendicular distance from a point M to a line to a line l
given by $\underline{\mathbf{r}} = \underline{\mathbf{a}} + t\underline{\mathbf{b}}$



Let $\underline{\mathbf{n}}$ be a vector joining M to the line l such that $\underline{\mathbf{n}}$ is perpendicular to M. Then, if we can find $\underline{\mathbf{n}}$ the required length will be $|\underline{\mathbf{n}}|$ that is, the length of $\underline{\mathbf{n}}$.

Since $\underline{\mathbf{n}}$ joins M to the line,

$\underline{\mathbf{r}} = \underline{\mathbf{m}} + \underline{\mathbf{n}}$

satisfies the vector equation of the line Where $\underline{\mathbf{r}}$ is the position vector
of the point N, where $\underline{\mathbf{n}}$ meets l where,

$\underline{\mathbf{n}} = \underline{\mathbf{r}} - \underline{\mathbf{m}} = \underline{\mathbf{a}} + t\underline{\mathbf{b}} - \underline{\mathbf{m}}$

But $\underline{\mathbf{n}}$ is also perpendicular to the line, hence
$\underline{\mathbf{n}} \cdot \underline{\mathbf{b}} = 0$
$\therefore \left(\underline{\mathbf{a}} + t\underline{\mathbf{b}} - \underline{\mathbf{m}}\right) \cdot \underline{\mathbf{b}} = 0$

This equation can be used to find t, which will give the position vector $\underline{\mathbf{r}}$ from
the point N

Hence, knowing $\underline{\mathbf{r}}$ we can find $\underline{\mathbf{n}}$.

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Example

To illustrate this; find the perdendicular distance of the point $M = (2, -3, 3)$ from the line l given by

$$\underline{\mathbf{r}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Solution

We have $(\underline{\mathbf{a}} + t\underline{\mathbf{b}} - \underline{\mathbf{m}}) \cdot \underline{\mathbf{b}} = 0$

where $\underline{\mathbf{a}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\underline{\mathbf{b}} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$

On substituting

$$\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}\right\} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 0$$

$$\therefore \quad \left\{\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

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$$-2-2-2+t(4+1+1)=0$$
$$-6+6t=0$$
$$t=1$$

$$\therefore\ \underline{\mathbf{r}}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}3\\-2\\2\end{pmatrix}$$

gives the point on the line l that is the point of intersection of the perpendicular from M to l.

Then $\underline{\mathbf{n}}=\begin{pmatrix}3\\-2\\2\end{pmatrix}-\begin{pmatrix}2\\-3\\3\end{pmatrix}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

is the perpendicular from M to l.

Thus the perpendicular distance is

$$\left|(1,1,\text{-}1)\right|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

In practice, remembering formulae like

$$(\underline{\mathbf{a}}+t\underline{\mathbf{b}}-\underline{\mathbf{m}})\cdot\underline{\mathbf{b}}=0$$

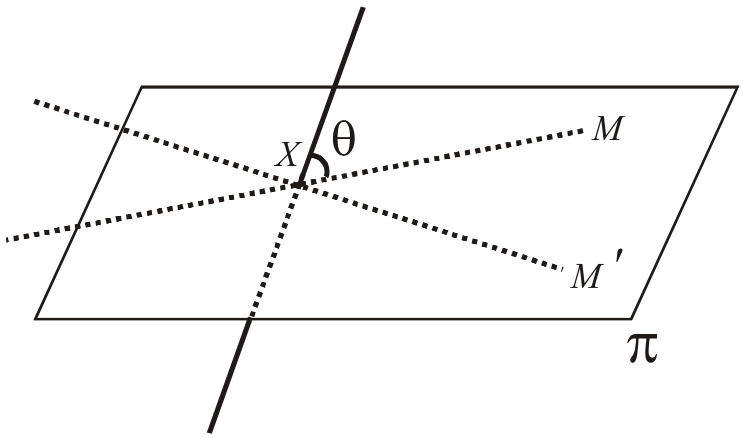
is not easy. The way to solve problems like this is to draw a diagram and work from the diagram to the solution.

(5) To find the angle between a line and a plane and the angle between two planes.

You have made a meal of this. Angle between line and plane is

$$\theta=\sin^{-1}\frac{\underline{\mathbf{u}}\cdot\underline{\mathbf{b}}}{|\underline{\mathbf{u}}||\underline{\mathbf{b}}|}$$

See Edexcel P4 p. 233



There could be many angles defined beween a line and a plane. In the above diagram, imagine rotating the point M (that lies in the plane) about the point *x*, which is the point of intesection of the line and the plane. As M is rotated the angle θ will also vary.

The angle between a line and a plane is defined to be the smallest angle of all the possible angles between the line and the plane.

If $\underline{\mathbf{u}}$ is the vector perpendicular to the plane then the angle between a plane and a line is complementary to the angle between the vector $\underline{\mathbf{u}}$ and the line $-$ that is, if we find the angle α between the normal to the plane and the line, then the required angle between the plane and the line is

$\theta = \pi/2 - \alpha$

(or in degrees $\theta = 90^\circ - \alpha$)

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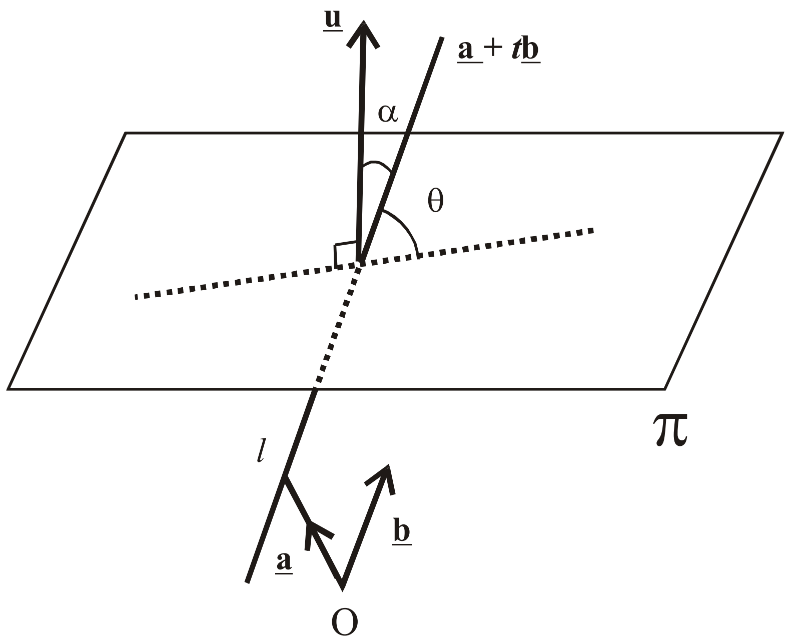
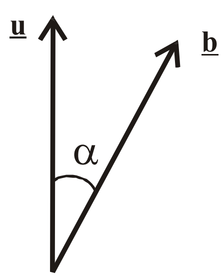


$\theta = \pi/2 - \alpha$



The angle between the normal $\underline{u}$ and the line l is given by

$$\cos\alpha = \frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{b}}}{|\underline{\mathbf{u}}||\underline{\mathbf{b}}|}$$

Thus, if we are given the vector form of the plane and the line finding the angle between them is almost automatic.

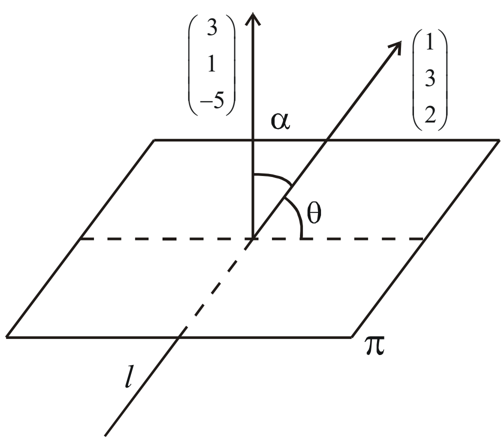
Example

Let π be the plane defined by $\underline{\mathbf{r}} \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = 6$

and l be the line defined by $\underline{\mathbf{r}} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

Find the cosine of the angle between them.

Answer



$$\cos\alpha = \frac{(3,1,-5)\cdot(1,3,2)}{|(3,1,-5)||(1,3,2)|}$$

$$= \frac{3+3-10}{\sqrt{9+1+25}\sqrt{1+9+4}}$$

$$= \frac{-4}{\sqrt{35}\sqrt{14}} = \frac{-4}{7\sqrt{10}}$$

The negative value of the cosine means that we have found the cosine of the obtuse angle between the vectors (3,1,-5) and (1,3,2). As we want the acute angle we drop the negative sign.

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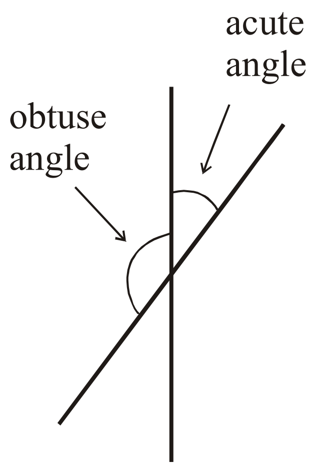
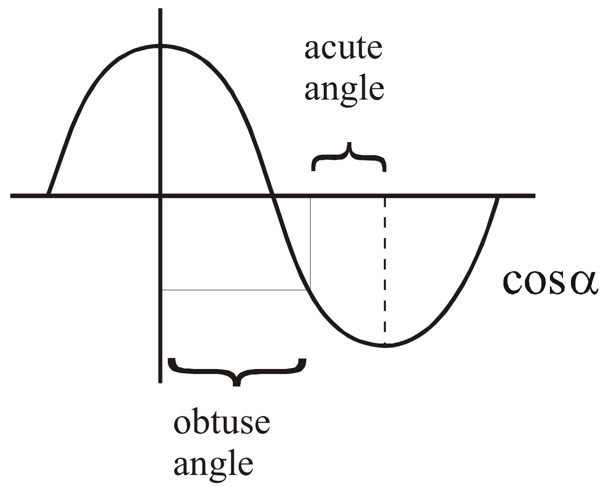
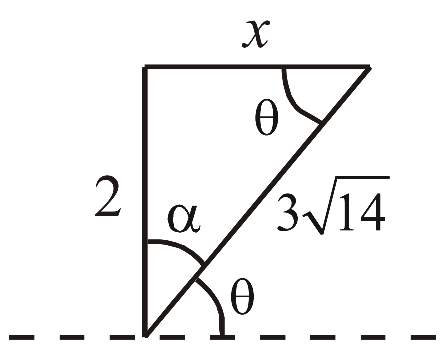




Then the required angle θ is found as follows



$x = \sqrt{9 \times 14 - 4} = \sqrt{122} = \sqrt{2}\sqrt{61}$

Hence $\cos\theta = \dfrac{\sqrt{2}\sqrt{61}}{3\sqrt{14}}$

The angle between two planes is equal to the angle between the two vectors normal to each plane respectively

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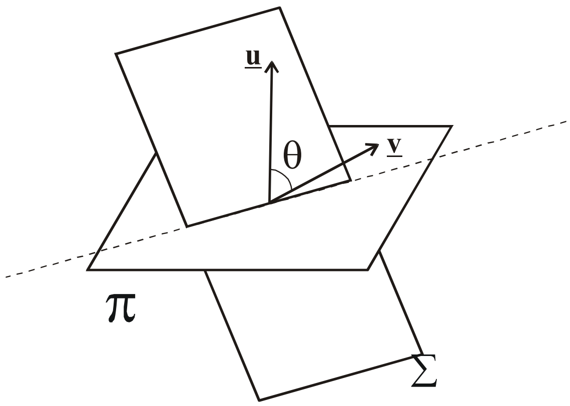


Let π be defined by $\underline{\mathbf{r}} \times \underline{\mathbf{u}} = k$ and Σ be defined by $\underline{\mathbf{r}} \times \underline{\mathbf{v}} = k'$; then the angle θ between π and Σ is the angle between $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$; that is:

$$\cos\theta = \frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{v}}}{|\underline{\mathbf{u}}||\underline{\mathbf{v}}|}$$

Example

Find the angle between the planes:

π $\quad \underline{\mathbf{r}} \times \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = 3$ and

Σ $\quad \underline{\mathbf{r}} \times \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 2$

Answer

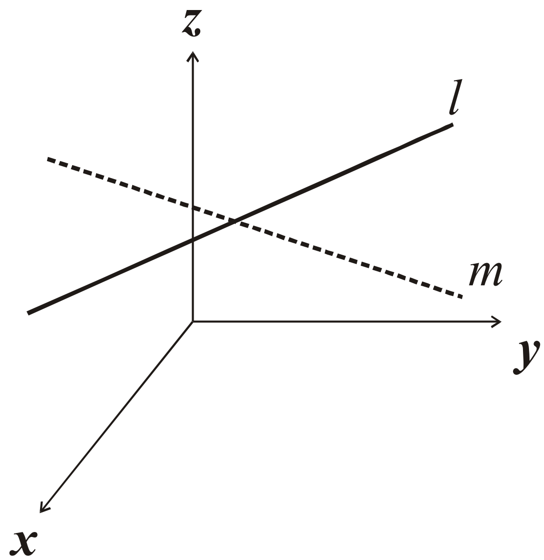
$$\cos\theta = \frac{(1,5,1).(-1,2,2)}{|(1,5,1)||(-1,2,2)|}$$

$$= \frac{-1+10+2}{\sqrt{1+25+1}\sqrt{1+4+4}} = \frac{11}{3\sqrt{26}}$$

$\therefore \theta = 44.0°$ (to 0.1°)

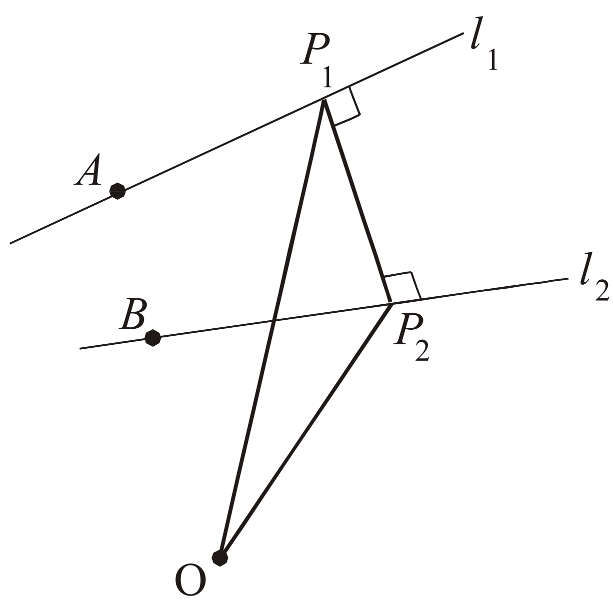
(6) To find the shortest distance between two skew lines.

In three dimensions two lines are skew when they are not parallel and do not intersect.



Two skew lines - the line *m* is in front of *l* and does not meet it.

Then the shortest distance between these two lines will be the length of the line that is perpendicular to both of them.



Let $\overrightarrow{P_1P_2}$ be the perpendicular vector joining l_1 to l_2 where l_1 is given by : $\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda\underline{\mathbf{p}}$ and l_2 is given by : $\underline{\mathbf{s}} = \underline{\mathbf{b}} + \mu\underline{\mathbf{q}}$.

Then P_1 lies on l_1 and P_2 lies on l_2. This enables us to find $\overrightarrow{P_1P_2}$ in terms of λ and μ. Then, since $\overrightarrow{P_1P_2}$ is perpendicular to both l_1 and l_2 we can use

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$\overrightarrow{P_1P_2} \cdot \underline{\mathbf{p}} = 0$

$\overrightarrow{P_1P_2} \cdot \underline{\mathbf{q}} = 0$

To find two simultaneous equations in λ and μ and solve to find P_1, P_2 and $\overrightarrow{P_1P_2}$

Example

For example, if l_1 is given by $\begin{pmatrix} 1 \\ 12 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$

and l_2 is given by $\begin{pmatrix} 9 \\ -12 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

find the shortest distance between them.

Answer

Let $P_1 = \begin{pmatrix} 1 \\ 12 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 9 \\ -12 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$$\text{Then } \overrightarrow{P_1P_2} = \left\{ \begin{pmatrix} 9 \\ -12 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\} - \left\{ \begin{pmatrix} 1 \\ 12 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} 8 \\ -24 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$$

$$\text{But } \overrightarrow{P_1P_2} \cdot \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 0$$

$$\therefore \left\{ \begin{pmatrix} 8 \\ -24 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right\} \cdot \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = 0$$

$30 + 2\mu + 19\lambda = 0$

$2\mu + 19\lambda = -30 \qquad (1)$

And $\overrightarrow{P_1P_2} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$

$$\therefore \left\{ \begin{pmatrix} 8 \\ -24 \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right\} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$\therefore -2 + 6\mu + 2\lambda = 0$

$6\mu + 2\lambda = 2 \qquad\qquad (2)$

Thus we have two simultaneous equations

$(1) \qquad 2\mu + 19\lambda = -30$

$(2) \qquad 6\mu + 2\lambda = 2$

$(1)\times 3$ gives $\qquad 6\mu + 57\lambda = -90 \qquad (3)$

$(3)-(2) \qquad 55\lambda = -92$

$$\lambda = -\frac{92}{55}$$

Substituting in (1)

$$2\mu - \frac{1748}{55} = -30$$

$$2\mu = \frac{98}{55}$$

$$\mu = \frac{49}{55}$$

Then $\overrightarrow{P_1P_2} = \begin{pmatrix} 8 \\ -24 \\ -6 \end{pmatrix} + \frac{49}{55}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \frac{92}{55}\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 262/55 \\ -1363/55 \\ -103/55 \end{pmatrix}$

Hence $\overrightarrow{P_1P_2} = \sqrt{\left(\frac{262}{55}\right)^2 + \left(\frac{1363}{55}\right)^2 + \left(\frac{103}{55}\right)^1} = 25.30(2.\text{d.p})$

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