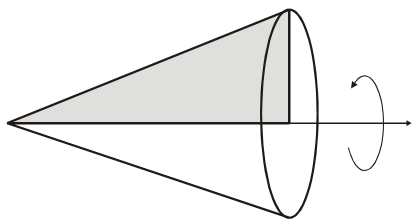
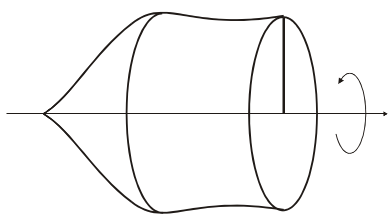
Volumes of Revolution

Volume of Revolution about the x-axis

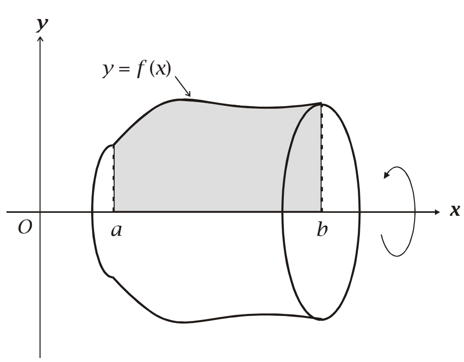
If a triangle is rotated about one of its sides a cone is generated.



The rotation of any area about an axis of symmetry results in a *solid of revolution* whose volume is a *volume of revolution*



Suppose the area in question is the area under the curve $y = f(x)$ between the limits $x = a$ and $x = b$. In other words it is the integral $I = \int_a^b f(x)dx$. Then the volume of revolution generated when this curve is rotated about the *x*-axis is $V = \pi\int_a^b y^2 dx$.



Example (1)

Find the volume of revolution generated when $y = 2x^2$ is rotated about the *x*-axis between $x = 1$ and $x = 3$.

Solution

$$y = f(x) = 2x^2$$

$$V = \int_a^b \pi y^2 dx$$

$$= \int_1^3 \pi \left(2x^2\right)^2 dx$$

$$= \pi \int_1^3 4x^4 dx$$

$$= \pi \left[\tfrac{4}{5}x^5\right]_1^3$$

$$= \pi\left(\frac{972}{5} - \frac{4}{5}\right) = \frac{968}{5}\pi$$

Example (2)

The region bounded by the curve $\sqrt{x} + \frac{1}{5}x$, the *x*-axis and the lines $x = 1$ and $x = 4$ is rotated through four right angles about the *x*-axis. Find, correct to one decimal place, the volume of the solid formed.

Solution

$$y = \sqrt{x} + \frac{1}{5}x$$

$$y^2 = \left(\sqrt{x} + \frac{1}{5}x\right)^2 = x + \frac{2}{5}x^{\frac{3}{2}} + \frac{1}{25}x^2$$

$$V = \pi \int_1^4 y^2 dx$$

$$= \pi \int_1^4 \left(x + \frac{2}{5}x^{\frac{3}{2}} + \frac{1}{25}x^2\right) dx$$

$$= \pi \left[\frac{1}{2}x^2 + \frac{4}{25}x^{\frac{5}{2}} + \frac{1}{75}x^3\right]_1^4$$

$$= \pi\left[\left(8 + \frac{128}{25} + \frac{64}{75}\right) - \left(\frac{1}{2} + \frac{4}{25} + \frac{1}{75}\right)\right]$$

$$= 13\frac{3}{10}\pi$$

$$= 41.8 \text{ (1 d.p.)}$$

Proof of this formula

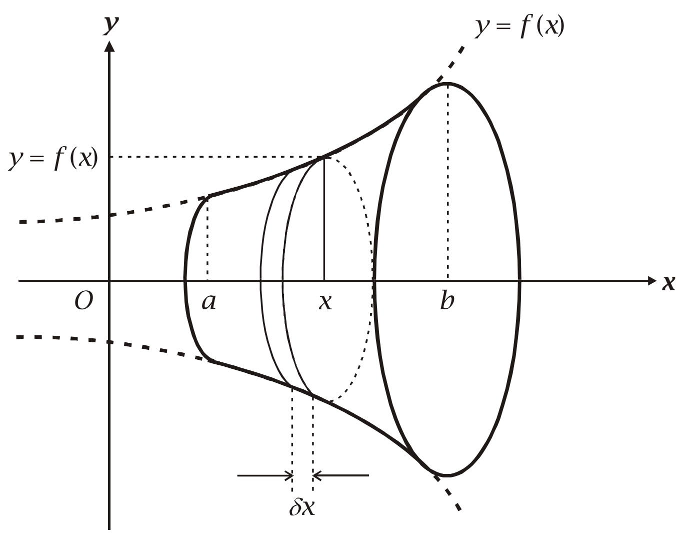
This section is optional

To prove

Let $I=\int_a^b f\left(x\right)dx$ be the area under the curve $y=f\left(x\right)$ between the limits $x=a$ and $x=b$. Then the volume of revolution generated when this area is rotated about the *x*-axis is $V=\pi\int_a^b y^2dx$.

Proof

Divide the volume into cylinders of width δx



The radius of each cylinder is $y=f\left(x\right)$. The area of a disc at x is πy^2. The volume of a cylinder is the product of this area and the width, δx .

Volume of cylinder $=\pi y^2\times\delta x$

The volume of revolution is approximated by the sum of all these discs. Let $\sum$ denote the taking of a sum by. Then

volume of revolution, $V\approx\sum\pi y^2\delta x$ between the limits $x=a$ and $x=b$.

(This says the volume of revolution is the sum of the volumes of all the cylinders between the given limits.) As the width of the cylinders gets smaller and smaller this approximation gets closer and closer to the true volume. In the limit, as $\delta x\to 0$, (as the number of cylinders $n\to\infty$) the approximation becomes the exact volume. The limit of $\sum\pi y^2\delta x$ is the integral $V=\pi\int_a^b y^2dx$.