Wilcoxon signed rank test on a population median

This is a non-parametric test to test a hypothesis about a population median. As with so many of these non-parametric tests it is best introduced through a worked example.

Example 1

In a Spanish town each year there is an annual bull festival. Young men run from one end of the town to another being chased by a bull. The young men regularly train and a scientifically minded elderly lady has observed that the median time for running across town when not being chased by a bull is 12 minutes. The same lady observed the bull festival from the same distance and noted the following times for running across town from a random sample of eight men:

10.3 9.6 8.4 14.1 12.3 7.2 6.9 9.8

By means of a single-sample Wilcoxon signed rank test, test the hypothesis that the median time for running across town is 12 minutes regardless of whether you are chased by a bull or not. Why might a t-test for the mean running time not be appropriate?

The hypotheses are:

H_0 : median time $= 12$

H_1 : median time < 12

This is a one-tailed test, $\alpha = 0.05$.

We begin by calculating the differences of the sample values from the expected median; we then rank the absolute values of the differences according to the magnitudes – from smallest to largest. We calculate the signed rank.

n	X	$d = X - 12$	rank	sign	signed rank
1	10.3	-1.7	2	-	-2
2	9.6	-2.4	5	-	-5
3	8.4	-3.6	6	-	-6
4	14.1	2.1	3	+	+3
5	12.3	0.3	1	+	+1
6	7.2	-4.8	7	-	-7
7	6.9	-5.1	8	-	-8
8	9.8	-2.2	4	-	-4

Let T_+ denote the sum of the ranks of the positive differences, and T_- denote the sum of the negative differences.

Then

$T_+ = 3 + 1 = 4$

$$\frac{n\cdot(n+1)}{2} - T_+ = \frac{1}{2}\cdot 8\cdot 9 - 4 = 32$$

The test statistic can be either T_+ or T_-. If T_+ is the small (and T_- is large) then the positive differences are small compared with the negatives ones. This is pushing us towards acceptance of the alternative hypothesis

H_1 : median time < 12 .

Thus, in this case the critical region is

$$T_+ \le T_{critical} \qquad \left(\text{or } T_- \ge \frac{n\cdot(n+1)}{2} - T_{critical}\right)$$

where the critical value is drawn from the distribution of T_+. This is given as Wilcoxon's *T*. Tables give critical values as a function of the sample size, the level of significance and whether the test is one- or two-tailed.

Here $T_{critical} = 5 \quad (n = 8,\ \alpha = 0.05,\ \text{one-tailed})$.

Since $T_+ < T_{critical} \qquad \left(\text{or } \ T_- > \frac{n\cdot(n+1)}{2} - T_{critcal}\right)$

we reject H_0 and accept H_1.

Being chased by a bull really does improve your ability to run.

The *t*-test in situation like the one here night not be appropriate because we can not guarantee the normality of the underlying population. The Wilcoxon signed rank test needs a much weaker assumption, namely, that the underlying population has a continuous probability distribution.

The next example shows how to deal with a two-sided alternative hypothesis.

<u>Example 2</u>

The median of the average monthly sales of an agent in an insurance company is 23. In a certain town the director of the local office of the insurance company recorded the average monthly sales of 10 randomly chosen agents.

The results were

30.3 17.3 33.9 37.0 20.6 35.9 20.1 29.7 24.7 26.8 .

By means of a single-sample Wilcoxon signed rank test, test whether or not the median of the average monthly sales in this particular town differs from 23. Use the 5% level of significance.

Solution

The hypotheses are

H_0 : median $= 23$
H_1 : median $\neq 23$.

This is a two-tailed test, $\alpha = 0.05$.

Agent	X	$d = X - 23$	rank	sign	signed rank
1	30.3	7.3	8	+	+3
2	17.3	-5.7	6	-	-6
3	33.9	10.9	9	+	+9
4	37.0	4.0	5	+	+4
5	20.6	-2.4	2	-	-2
6	35.9	12.9	10	+	+12
7	20.1	-2.9	3	-	-3
8	29.7	6.7	7	+	+7
9	24.7	1.7	1	+	+1
10	26.8	3.8	4	+	+4

$T_- = 6 + 2 + 3 = 11$

$T_+ = \frac{1}{2} \cdot 10 \cdot 11 - 11 = 44.$

As in the single-sample case, the test statistic can be either T_+ or T_-. In the case of the two-tailed test, both the large and small values of T_+ $\left(\text{or } T_-\right)$ should make us suspicious about the validity of the null hypothesis. The critical region is this case has the form

$$T_+ \leq T_{critical} \qquad \left(\text{or } T_- \geq \frac{n \cdot (n+1)}{2} - T_{critical} \right)$$

where T is either T_+ or T_-.

Here $T_{critical} = 8 \qquad \left(n = 10,\ \alpha = 0.05, \text{ two-tailed}\right)$.

Since $T_- > T_{critical} \qquad \left(\text{or} \quad T_+ < \frac{n \cdot (n+1)}{2} - T_{critcal} \right)$

We accept H_0 and reject H_1.

No evidence that the median of the average monthly sales differs from 23.