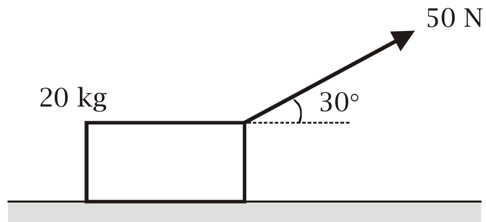
Work

Prerequisites

You should be familiar already with Newton's second law, with friction and with problems involving the resolution of forces horizontally and vertically.

Example (1)



The diagram shows a block of mass 20 kg in contact with a horizontal surface. The block is initially at rest when at time $t = 0$ it is pulled by a constant force of 50 N by means of tension in a rope attached to the block making an angle of 30° to the horizontal. This constant force is maintained for 10 seconds, after which the force is removed. The coefficient of friction between the block and the surface is 0.2. Assume that the entire under surface remains in contact with the block, so that the contact force between the block and the surface also remains constant.

(*a*) Find the resultant force acting on the block in a horizontal direction.

(*b*) Find the acceleration of the block during the 10 seconds that it is subject to the pulling force of 50 N.

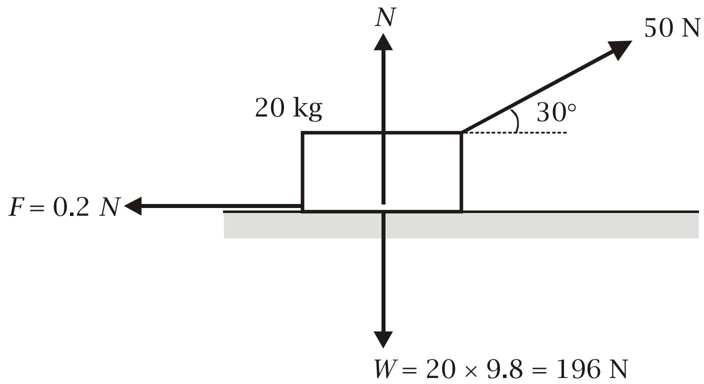
(*b*) Find the distance travelled by the block during the first 10 seconds of its motion.

(*d*) Find also the velocity of the block at 10 seconds.

(*e*) Find the distance travelled by the block once the pulling force of 50 N is removed.

Solution

(*a*) There are four forces acting on the block during the period when it is subject to the force of 50 N.



These are the 45 N force acting at an angle of 30° degrees to the horizontal, the weight of the block $(W = mg = 20 \times 9.8 = 196\text{N})$, the normal reaction (N) and the friction $(F = \mu N = 0.2N)$. Let the resultant horizontal force be R.

(*a*) Resolving horizontally and vertically

$(\uparrow)$ $N + 50\sin 30° = W$

$N = 20 \times 9.8 - 50 \times 0.5 = 171$

$(\rightarrow)$ $R = 50\cos 30° - F$

$$= 50 \times \frac{\sqrt{3}}{2} - 0.2 \times 171$$
$$= 9.101...$$
$$= 9.10 \text{ N } (3 \text{ s.f.})$$

(*b*) Applying Newton's second law

$$a = \frac{R}{m} = \frac{9.101...}{20} = 0.455 \text{ ms}^{-2} \ (3 \text{ s.f.})$$

(*c*) This requires one of the equations of uniform acceleration. The initial velocity is $u = 0$. Hence

$$s = ut + \frac{1}{2}at^2 = \frac{1}{2} \times 0.455... \times (10)^2 = 22.8 \text{ m } (3 \text{ s.f.})$$

(*d*) This is another equation of uniform acceleration. Again $u = 0$.

$$v = u + at = 0.455... \times 10 = 4.55 \text{ ms}^{-1}$$

(*e*) Once the 50 N force is removed the block has an initial velocity of $u = 4.55 \text{ ms}^{-1}$. It is subject to a retarding force of friction. This is given by

$$F = \mu N = \mu mg = 0.2 \times 20 \times 9.8 = 39.2 \text{ N}$$

The deceleration is

$$a = \frac{39.2}{20} = 1.96 \text{ ms}^{-2}$$

The final velocity is $v = 0$. We require another equation of uniform acceleration.

$$v^2 = u^2 + 2as$$
$$0 = (4.55...)^2 - 2 \times 1.96s$$
$$s = \frac{(4.55...)^2}{2 \times 1.96} = 5.28... = 5.28 \text{ m} \ (3 \text{ s.f.})$$

This example revises the essential background concepts for this chapter and leads naturally to the topic of work.

Work

Let us continue with example (1). Imagine that you were at the end of the rope pulling the block. How would you feel during the whole 10 seconds experience? To answer this subjective question – you would be *working*. The effort you put into pulling the block would require an *effort* on your part – that is, an expenditure of energy. But from the mathematical point-of-view the subjective feelings that accompany *work* are irrelevant. Work is done whenever a force acts on an object over a certain distance. When people do work they experience fatigue, but the forces themselves are impersonal. In example (1) when the 50 N force is removed the object is slowed down by friction, then the frictional force also does work though no human effort is required.

Mathematicians seek a precise definition of work, and one that does not rely on subjective statements. For them *work* is done when a force makes an object move. The mathematical definition of work is

work = force $\times$ distance   work = Fd

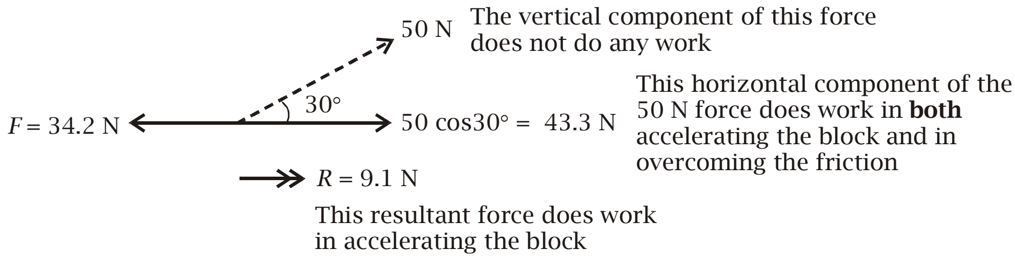
Work is a scalar quantity. The units of work are joules (symbol J).

Example (2)

A force of $14N$ moves an object through a distance of 6 m. Find the work done.

$$\begin{aligned} \text{work} &= Fd \\ &= 14 \times 6 = 84 \text{ J} \end{aligned}$$

The distance must be measured in the direction of the motion of the object. In example (1) the force to 50 N acts at an angle of 30° to the motion of the object. Therefore, the force that acts on the block and does work is not 50 N but the horizontal component of this 50 N force, which is $50\cos 30° = 43.3$ N .



Furthermore, as the above diagram shows, in example (1) during the first part of the motion, this force of $50\cos 30°$ N in fact does **two** forms of work. Firstly, it accelerates the block along the surface; secondly, it overcomes the frictional force that is resisting the motion.

Example (1) continued

In this part of the question give your answers to 2 significant figures.

(*f*) Find the work done by the horizontal component of the 50 N force during the first 10 seconds of the motion of the block.

(*g*) Find the work done in accelerating the block during the first 10 seconds.

(*h*) Find the work done by friction when the block slows down after 10 seconds.

(*i*) Comment on your answers to parts (*g*) and (*h*).

(*j*) What eventually happens to the work that is done by overcoming friction in the 10 seconds of the motion of the block and the work subsequently done by friction in slowing the block down?

Solution

(*f*) The horizontal component of the force is

$50\cos 30° = 43.3$ N

During the given time period this moves through a distance of 22.8 m.

$$\begin{aligned}\text{work} &= Fd \\ &= 43.3 \times 22.8 = 990 \text{ J } (2 \text{ s.f.})\end{aligned}$$

(*g*) In part (*a*) we found the resultant force to be 9.10 N. In part (*c*) we found that this resultant force acts on the block over a distance of 28.8 m. Hence

$$\begin{aligned}\text{work} &= Fd \\ &= 9.10 \times 22.8 = 210 \text{ J } (2 \text{ s.f.})\end{aligned}$$

(*h*) The friction is 39.2 N and the distance is 5.28 m. Hence

$$\begin{aligned}\text{work} &= Fd \\ &= 39.2 \times 5.28 = 210 \text{ J } (2 \text{ s.f.})\end{aligned}$$

(*i*) To the level of accuracy given in the solutions, the work done in accelerating the block is equal to the work done in slowing it down (210 J). We assert that in fact the two values are strictly equal and we shall justify this below.

(*j*) All work against friction is eventually converted into heat.

In the solution to part (*i*) above we claim that the work required to accelerate an object from rest to a certain speed $\left(v \text{ ms}^{-1}\right)$ is the same as the work required to decelerate it back to rest. We can prove this.

Example (3)

A particle of mass m kg is subject to a (resultant) force F newtons that causes it to move d metres in the direction of F. At the end of d metres, the particle has speed $v \text{ ms}^{-1}$. Show that the energy of the particle at the end of d metres is given by $\frac{1}{2}mv^2$. Hence prove that the work done in accelerating a particle to $v \text{ ms}^{-1}$ is the same as the work done in decelerating it. Hint: use one of the equations of uniform acceleration to find v and use the definition of work as the product of force and distance to show that the work is $\frac{1}{2}mv^2$.

Solution

Following the hint, firstly the acceleration of the particle is

$$a = \frac{F}{m}$$

Substituting into $v^2 = u^2 + 2as$ where u is the initial velocity given by $u = 0$

$$v^2 = u^2 + 2as$$

$$v^2 = 2\frac{F}{m}d$$

Rearranging this equation, the work done on the particle by the force F is

$$\text{work} = Fd = \frac{1}{2}mv^2$$

Regarding the second part of the question, we first observe that in this equation work is given by the product $\text{work} = Fd$ and the precise magnitude of the force or distance is not specified. This means that if a smaller force is given then the same work $\left(\frac{1}{2}mv^2\right)$ can be done over a longer distance. Secondly, the quantity v^2 is always positive (being a square). So if the signs are all reversed (equivalent to decelerating the particle) then the work done is still the same.

When work is done energy is converted. When pulling the block along the level surface the energy in your muscles is converted into the work done in accelerating the object and in overcoming friction. When lifting a crate the energy in your muscles is converted into the work done against gravity, which is the same as the increase in the crate's gravitational potential energy. So we have

work = change in energy

This means that from the physicist's point-of-view work is another term for energy, and as a consequence of this, their units are the same – and energy is also measured in joules.

The units of energy

Energy is measured in joules (symbol J). However, the equation

$$\text{work} = \text{force} \times \text{distance}$$
$$= \text{mass} \times \text{acceleration} \times \text{distance}$$

shows that the units of energy can also be quoted as

$$\text{units of energy} = \text{kg} \times \text{ms}^{-2} \times \text{m} = \text{kg}\,\text{m}^2\text{s}^{-2}$$

These are the units of energy in terms of the fundamental units of mass (kg), length (m) and time (s).

In example (1) the block acquires energy in respect of its motion. The solution to example (3) shows that the work done on the block to give it this energy is

$$\text{work} = Fd = \frac{1}{2}mv^2$$

By the principle that

work = change in energy

this means that the energy of the block because it is moving is also $\frac{1}{2}mv^2$. This energy is called *kinetic energy*. Thus

$$\text{kinetic energy} = \frac{1}{2}mv^2$$

Physicists believe that whenever work is done one form of energy is converted into another. Furthermore, in this process, no energy is lost. This is in accordance with the principle of conservation of energy.

Principle of conservation of energy

Total energy in the universe is neither created nor destroyed, but only converted from one form to another.

This is a very general statement of the principle of conservation of energy. In practice we are concerned with conversions between one form of energy and other forms. In this case, where one form of energy is gained and the other is lost the energy gained must be equal to the energy lost. In example (1) when the block was being accelerated it gained kinetic energy. In the second part it lost kinetic energy, which was converted by friction to heat. A system of energy conversions where total energy remains the same is called a *closed system.*

Working against gravity

Another example of work would be when lifting objects. Imagine carrying a crate of goods up a staircase. In this case you are *working against gravity.* As you take the crate up the stairs the crate acquires *gravitational potential energy.* If you drop the crate and it falls down the stairs the crate loses this energy. When an object is in free fall it is losing its gravitational potential energy, which is being converted to kinetic energy – the energy of a moving object. The force of gravity causes the object to fall through a certain distance; it also causes it to accelerate. As the object's speed increases its kinetic energy increases.

The gravitational potential energy (U) acquired by an object when it is raised by a height (h) is an instance of the definition of work. Compare

work = force $\times$ distance $\quad$ work = Fd

with

increaase in gravitational potential energy = weight $\times$ height

You will see that the second equation is in fact an instance of the first. Weight is a force, height is a distance, and the increase in gravitational potential energy is the work done against gravity. Because the weight is given by

weight = mass $\times$ acceleration due to gravity $\quad (W = mg)$

we see that the gravitational potential energy of an object is given by

gravitational potential energy = mass $\times$ acceleration due to gravity $\times$ height $\quad U = mgh$

For objects in free fall the kinetic energy gained by the object is equal to the gravitational potential energy it loses.

gain of kinetic energy = loss of gravitational potential energy

This leads to the equation

$$\frac{1}{2}mv^2 = mgh$$

We see that we can cancel through the mass (m) to obtain

$$v^2 = 2gh$$

This equation governs the energy conversions that take place between kinetic energy and gravitational potential energy.

Example (4)

An object falls 100m. Ignoring air-resistance, calculate its terminal velocity.

Solution

Loss of gravitational potential energy = gain of kinetic energy. Therefore

$$mgh = \frac{1}{2}mv^2$$

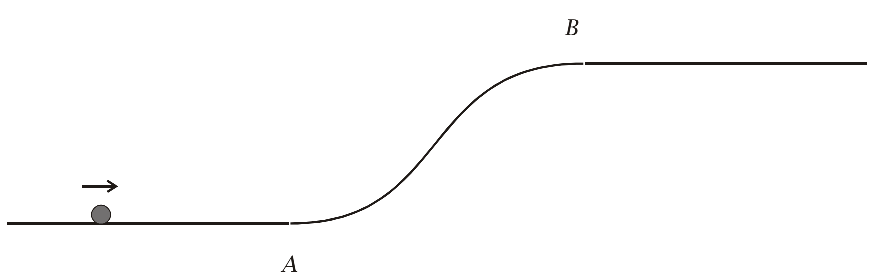
$$v^2 = 2gh$$

$$\begin{aligned} v &= \sqrt{2gh} \\ &= \sqrt{2 \times 9.8 \times 100} \\ &= 44.271 = 44.3 \text{ ms}^{-1} \ (3 \text{ s.f.}) \end{aligned}$$

Whenever an object loses height it gains kinetic energy. In reverse, when a moving object gains height it loses kinetic energy.

Example (5)

The diagram shows a particle *P* of mass 1 kg moving along a frictionless track



The point *A* lies at the foot of the track and the point *B* at the top. The vertical distance between *A* and *B* is 3.6 m. At *A* the particle has speed 20.5 ms^{-1}. Find the speed of *P* when it reaches *B*.

Solution

The kinetic energy lost by the particle must be equal to the gravitational potential energy it gains. Because the track is frictionless no other energy conversions need to be considered. The kinetic energy of the particle at *A* is

$$K = \frac{1}{2}mv^2 = \frac{1}{2} \times 1 \times (20.5)^2 = 210.125 \text{ J}$$

At *B* the gain of gravitational potential energy is

$$U = mgh = 1 \times 9.8 \times 3.6 = 35.28 \text{ J}$$

This gain in gravitational potential energy is equal to the loss of kinetic energy of the particle. So its kinetic energy at B is

kinetic energy $= 210.125 - 35.28 = 174.845$ J

Let u denote the speed of the particle at B. Then

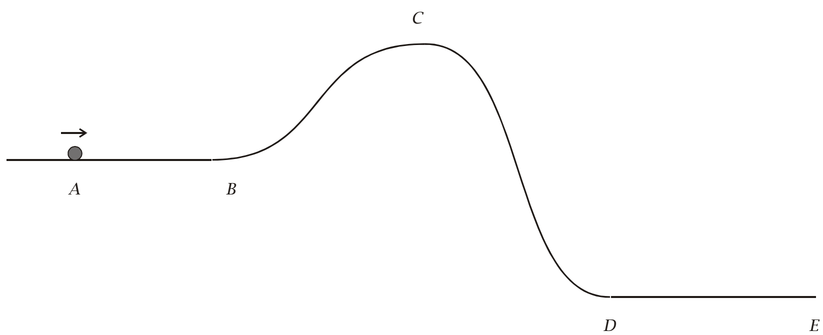
$$\frac{1}{2}u^2 = 174.845$$

$$u = \sqrt{349.69} = 18.7 \text{ ms}^{-1}$$

In example (5) we allowed the simplifying assumption that the track was frictionless. However, we can also introduce friction into the question. Providing we can determine the *work done against friction* we can still use the principle of conservation of energy to make deductions.

Example (6)

The diagram shows a particle P of mass 5 kg moving along a track



The sections AB and CD are level. After passing over C the particle descends to D and thereafter moves along a final horizontal portion of the track to pass through E. The vertical distance between B and C is 5 m, and the height of C above D is 9 m. At A the particle has speed 18 ms^{-1}. The length of the track between A and B is 10 m, between B and C it is 12 m, between C and D it is 24 m, and between D and E it is 15 m. Throughout the journey the motion of the particle is resisted by a constant force of magnitude 6 N. Find the speed of P when it reaches E. You may assume that the particle has sufficient energy to reach the point C.

Solution

There is a slight "trick" in this question, since the distances for all the segments are given. However, we require only the total length of the track between A and E, which is

length of track = $10 + 12 + 24 + 15 = 61$ m

This is the distance over which the frictional force of 6 N works.

So the energy lost by the particle P in moving between A and E due to friction is

$\text{work} = Fd = 6 \times 61 = 366 \text{ N}$.

The energy of the particle at A

$$K_A = \frac{1}{2}mv^2 = \frac{1}{2} \times 5 \times (18)^2 = 810 \text{ J}$$

Provided the particle has sufficient energy to get over the hillside at C (which we are told to assume) then to calculate its energy at E what we need to know is the vertical drop between A and E, which is

$$h = 9 - 5 = 4 \text{ m}$$

In descending from the level AB to the level DE the particle loses gravitational potential energy equivalent to this drop.

$$U = mgh = 5 \times 9.8 \times 4 = 196 \text{ J}$$

So at E the energy of the particle is

$$\begin{aligned} \text{energy at } E &= \text{energy at } A + \text{loss of potential} - \text{work against friction} \\ &= 810 + 196 - 366 \\ &= 640 \end{aligned}$$

Let u denote the speed of the particle at E. Then

$$\frac{1}{2}mu^2 = 640$$

$$\frac{1}{2} \times 5 \times u^2 = 640$$

$$u = \sqrt{256} = 16 \text{ ms}^{-1}$$

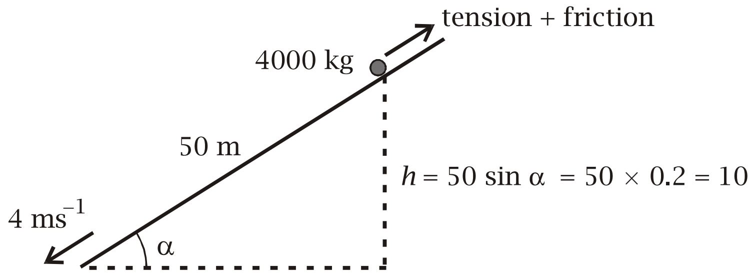
In this question we are given the frictional force and work from that to the final velocity of a particle. Problems can be set in reverse.

Example (7)

A straight track is inclined at an angle of α to the horizontal, where $\sin\alpha = 0.2$. The motion of a wagon down the track is controlled by means of a cable, which is parallel to the track. The tension in the cable is 5000 N. The wagon, of mass 4000 kg, starts from rest at a point on the track and after travelling 50 m, its speed is 4 ms^{-1}. The resistance to motion is R N, where R is a constant. Use energy considerations to calculate the value of R.

[Adapted from WJEC, June 2005 (legacy specification)]

Solution



As the wagon rolls down the track it gains kinetic energy.

Gain of kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2} \times 4000 \times 4^2 = 32000$ J

It also loses gravitational potential energy.

Loss of gravitatioal potential energy $= mgh$

$$= 4000 \times 9.8 \times 50 \sin\alpha$$
$$= 4000 \times 9.8 \times 50 \times 0.2$$
$$= 392000 \text{ J}$$

It also works against the tension in the cable.

Work done against the tension $= Fd = 5000 \times 50 = 250000$

It also works against the frictional resistance, R, which is unknown.

Work done against the resistance $= Rd = 50R$

By the principle of conservation of energy

Loss of potential = gain of kinetic energy + work against tension + work against resistance

$392000 = 32000 + 250000 + 50R$

$50R = 110000$

$R = 2200$ N