Poincaré's Thesis

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Abstract

This is a discussion paper. No part of this paper may be considered to be true until verified by competent authorities. This paper contains putative proofs of the following propositions: -

1. Proof of the validity of Poincaré's thesis that the principle of complete induction cannot be embedded into an effectively computable sub-domain of any model corresponding to a sufficiently strong first-order theory – that is, a theory which is an instance of formal analytic logic.

2. Solution to the Halting problem.

3. Proof that the identification within set theory of the set of all natural numbers, $\mathbb{N}$, with the set of all finite ordinals, ω, leads to contradiction.

4. Demonstration that the proof of Lagrange's theorem is not effectively computable.

5. Proof that transcendental numbers are defined by generic sequences.

6. Resolution of the Liar paradox.

7. Resolution of Berry's paradox.

8. Resolution of Grelling's paradox.

9. Demonstration that there exists a synthetic logic; that is, a logic that is not a formal analytic logic.

This paper also expounds a neo-Kantian philosophy of mathematics and advances a transcendental deduction to prove that no metaphysical conception of the mind based on the assumption that first-order set theory is effectively computable is adequate to account for human knowledge.

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CHAPTER 1

Poincaré's thesis

1 Strong AI and formalism

The claim that machines are intelligent is denoted by *AI*, which stands for *Artificial Intelligence*. "The objectives of AI are to imitate by means of machines, normally electronic ones, as much of human mental activity as possible, and perhaps eventually to improve upon human abilities in these respects." (Penrose [1989] p.14) The claim that machines can "think" as we humans do, are "intelligent" and "conscious" just as we are, is denoted by *strong AI*. "According to strong AI," as Penrose explains, "… mental qualities of a sort can be attributed to the logical functioning of *any* computation device, even the very simplest mechanical ones, such as a thermostat." (Penrose [1989] p.21)[1] By *effectively computable* or *recursive* is meant any process that is performed by a digital machine. The aim of this paper is to refute the doctrine of strong AI. The methodology of this paper is to achieve this by developing further mathematical insight into the nature of proof to show that no effectively computable process can match it.

In proofs we encounter axioms and rules. Traditionally, the axioms were regarded as primitive propositions apprehended by "the mind" by means of "intuition". Such a philosophy of mathematics does not cohere with Strong AI, which is consistent with a view of mathematics known as *formalism*. Formalism maintains that axioms may be mechanically formed in a language whose syntax can be recursively enumerated as a list; the *rules* are transformations of expressions of the formal language into other expressions. It is assumed that the operations of all rules and axioms thus given can be physically simulated in a machine.

> According to formalism the central concept in mathematics is that of a formal system. Such a system is defined by a set of conventions … we start with a list of elementary propositions, called *axioms*, which are true by definition, and then give *rules of procedure* by means of which further elementary theorems are derived. The proof of an elementary proposition then consists simply in showing that it satisfies the recursive definition of elementary theorem. (Curry [1954], p. 203)

[1] The term *strong AI* is due to John Searle. Other terms used in the literature are *functionalism* and *computationalism*.

I shall take this quotation as a definition of *formalism*.[2] Sometimes with formalist theories reference is made to *semantics* and in particular to *model theory*. Yet here there is no primitive notion deployed of *meaning, intension* or *concept*. The relation explored in effective formal semantics between a language and its model is conceived as another effective relation between one structure and another. In the formalist conception of semantics we don't break out of the formalism into anything other than more formalism. In the mathematics of formal systems there is only syntax: semantics is given by syntax. Curry, a nominalist, regards interpretation of the language of mathematics as irrelevant: "Why not abolish the object language altogether and understand that the tokens are objects which we can take as symbols if we want to?" (Curry [1954], p. 204) Such nominalism is an essential aspect of formalism and proponents of strong AI must adopt it.

2 Formalism and ZFC

Formalists require a working version of mathematics. Their favoured theory is known as ZFC, which stands for *Zermelo-Fraenkel set theory with Choice*. (See Chap. 2 / 1.3.4) This is also regarded as a *first-order theory*, meaning that it is embedded in a theory of formal logic known as the first-order predicate calculus. This preference is reflected ubiquitously in the literature: -

> Most logicians (though perhaps not most mathematicians) are convinced that all correct proofs in mathematics could, with enough effort, be translated into formal proofs of first-order logic. (Wolf [2005] p.29) … *ZFC* is a remarkable first-order theory. All of the results of contemporary mathematics can be expressed and proved within *ZFC*, with at most a handful of esoteric exceptions. Thus it provides the main support for the formalist position regarding the formalizability of mathematics. In fact, logicians tend to think of *ZFC* and mathematics as practically synonymous. (Wolf [2005] p.36)
>
> Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set-theoretic principles. From such axioms, all known mathematics may be derived. However, there are some questions which the axioms fail to settle, and that failure is the subject of this book. (Kunen [1980] p.xi.)

[2] Formalism in this sense should not be confused with the earlier philosophy of mathematics that originates in the work of Hilbert and which also is called *formalism*. Although a historical antecedent of the modern formalism of which Curry is a representative, this older philosophy has a very different foundation, and should be called *Hibertism*. I discuss it below [Chap. 16, Sec. 4].

> No arithmetical conjecture or problem that has occurred to mathematicians in a mathematical context, that is, outside the special field of logic and the foundations or philosophy of mathematics, has ever been proved to be undecidable in ZFC. (Franzén [2005] p.33)

There is an obvious immediate objection to formalism, which is that many of the statements and axioms of ZFC refer to entities that could not possibly be effectively computable. Examples are (1) the Axiom of Infinity, which states that an infinite collection exists, and (2) the Axiom of Choice, which states that it is possible to well order every collection whatsoever, even though no such order can be explicitly given. Formalists meet this objection with one of two strategies.

2.1 Constructivism

The first is to reject the use of those statements that refer to actually infinite collections. This philosophy is known as *constructivism*. "Any computation that can be performed by a finite intelligence – any computation that has a finite number of steps – is permissible." (Bishop [1967] p.3) An even more stringent version of this theory is *strict finitism*, which maintains that no infinitary operation is admissible in mathematics whatsoever. In this sense strict finitism does not allow any symbol for an infinite collection whatsoever: so by this token any mathematical formula employing the symbols ∞ or ω would be inadmissible in mathematics. This defence is discussed in Chapter 14 (Section 2.3).

2.2 Juggling finite sequences of symbols

The second, more favoured approach, is to treat axioms governing infinite objects as also mere formal symbols whose operations can be captured by mechanical processes.

> The Formalist can hedge his bets. The formal development of ZFC makes sense from a strictly finitistic point of view: the axioms of ZFC do not say anything, but are merely certain finite sequences of symbols. The assertion $ZFC \vdash \phi$ means that there is a certain kind of finite sequence of finite sequences of symbols – namely, a formal proof of ϕ. Even though ZFC contains infinitely many axioms, notions like $ZFC \vdash \phi$ will make sense, since one can recognize when a particular sentence is an axiom of ZFC. A Formalist can thus do his mathematics just like a Platonist[3], but if challenged about the validity of handling infinite objects, he can reply that all he is really doing is juggling finite sequences of symbols." (Kunen [1980] p. 7)

[3] Platonism in the philosophy of mathematics is the doctrine that mathematics refers to abstract entities and properties existing in a supra-sensible objective reality.

As the quotation indicates, this *juggling symbols* defence represents a second type strict finitism combined with nominalism[4]: in this view the symbols themselves have no meaning and all mathematics is a form of "juggling finite sequences of symbols". A property is said to be *decidable* if there is some effective procedure which, given any object, produces the answer *yes* if the object does have the property and *no* if not. The following comes from a book entitled *Computable Set Theory*; it shows how ambiguous the status of ZFC in this juggling symbols defence is: -

> We are interested in methods that could be automated, at least in principle; <u>it is hence obvious that none of our methods will be applicable to full set theory</u>. It is in fact known that no variant of set theory that deserves this name is decidable. (Cantone et al. [1989] p.13)

My underlining. It would appear that there is both a computable set theory and a non-computable set theory. We must, therefore, guard against a possible conflation of meanings attached to the word "decidable". It is true that proofs decide theorems, but in the context of effective computability, "decidable" means there exists an effectively computable procedure; it does not mean, "there is a formal proof". The prima facie evidence, as indicated by the above quotation, is that in general *formal proof in first-order set theory is not effectively computable.* Furthermore, if not all of ZF (or ZFC) is effectively computable, and the claim that all of known mathematics can be written in ZFC, then it follows: *not all of known mathematics is effectively computable.* So I conclude proponents of strong AI must believe that if ZFC formalizes any known part of mathematics then that known part of mathematics is thereby effectively computable. There is a sense in which all of known mathematics is trivially effective.

2.3 The glorified typewriter

1. Scan the known result together with its valid proof into a computer.
   This converts the characters of the book to Ascii code (or some such) and thence into machine code.
2. Click on the scan and print.
   This converts the machine code back into intelligible language.

The computer has produced a statement of the theorem together with its valid proof. I shall call this procedure the *glorified typewriter.*

Nobody does suggest that a *glorified typewriter* has ever checked or proven any theorem. However, this trivial procedure acts as the lower limit on what shall <u>not</u> count as a demonstration that mathematics is effectively computable. We need a test or criterion of what shall count. (The Turing Test is discussed below [Chap.13, Sec.1].) For the present, let us examine the position of theorem checkers in relation to the glorified typewriter.

[4] Nominalism denies the existence of abstract objects and universals to serve as the referents of general terms. In the philosophy of mathematics nominalism is diametrically opposed to Platonism.

> The task of a proof verifier is not to discover proofs (except proofs of the elementary individual steps within larger, more challenging proofs). Indeed, the mathematician or computer scientist using such a verifier will always know the proof of any theorem that they wish the verifier to accept. The problem is that, as known to the system user, this proof ordinarily has far too many missing details, intuitive leaps justified by geometric or other relatively informal insight, steps to be handled by analogy with other arguments given earlier or 'well-known in the literature', etc., for a computerized verifier to follow what is intended. Thus, in designing a proof verifier, one must provide some means for the formalization of this layer of missing detail. (Cantone et al. [1989] p. 3)

The underlining is my own. It is clear from this that a proof checker falls well below the bar required to demonstrate that mathematics is effectively computable. Proof checkers are sophisticated versions of the *glorified typewriter* or perhaps *spell-checker*. The *symbols are finite objects* but what they denote often is not. For example, suppose I read in a text the following: -

> ... *the least nonrecursive ordinal* ω_1^c is the recursive analogue of ω_1, the first-uncountable ordinal (Barwise [1975] p. 2)

I see the symbols ω_1^c and ω_1 both of which denote structures that by their very nature could not be an input of any actual digital machine – for they denote objects that are non-recursive and infinite. But as symbols they are finite and may have some Ascii or machine code attached to them. Therefore, all theorem-checkers that represent non-recursive and non-finite objects operate on the finite symbols or their machine code equivalents and not on the objects that they denote. They operate forms of equation logic and are *simulations of first-order set theory* rather than first-order set theory itself. Hence, if a theorem-checker produces a valid proof of a theorem that is no conclusive evidence to the effect that the theorem itself is effective. Since the class of decidable theorems is always admitted to be less than the class of all known theorems, there is no reason to suppose that a theorem checker is much more than a glorified typewriter.

2.4 Computational speed

Computers are useful aids to combinatorial problems – that is, calculations. I certainly would not challenge my pocket calculator to a competition as to which of "us" can multiply six digit numbers faster; but the fact that my calculator is much better at this than I am does not encourage me to conclude that it can think. A problem that depends on speed of computation is likely to be more solvable by a computer than by a man, precisely because *understanding* is not involved. For this reason, too, a computer in combination with human intelligence could be a more powerful theorem builder than either alone – at least, in problems that rely on a mixture of insight (provided by the humans) and computational muscle (provided by the machines). This is illustrated by the four-colour problem[5]. Michael Beeson in *The Mechanization of Mathematics* (Beeson [2003] p.) describes *Robbins conjecture*[6] “rediscovered” and proven by the computer program EQP. It lies well below the bar required to simulate mathematical creativity; it is a problem in equation logic that depends on finite combinatorics.

3 Prima facie argument against formalism

There are difficulties too with the statement: “It is an empirical fact that all of mathematics presently known can be formalized within the ZFC system”[7]. There is a problem with this claim as to an “empirical” fact and its validity may be seriously doubted. *I have never read a single mathematical text that is written in first-order set theory* (ZF or ZFC). As an empirical fact this claim appears to be refuted by every textbook, monograph, paper and communication, written or verbal of every mathematician, living or dead! It *may be* that some of these books are about structures that are defined in first-order set theory, but *the books themselves are not written in that theory.* In these books it is the meta-text, the gloss, that *is the mathematical text*, which is written in a mixture of second-order logic and natural language. Not only is the gloss that is written in this way, but the bulk of the proofs too. Consider the following theorem – to which I append one line taken from the proof: -

3.1 Theorem

If $f:[a,b]\to\mathbb{R}$ is Riemann integrable over $I=[a,b]$, then there exists a sequence of partitions $\{P_k\}$, $P_k\in\Pi$, such that P_{k+1} is a refinement of P_k, $k=1,2,\ldots$, and $\lim_{k\to\infty}\overline{S}_{P_k}(f)=\lim_{k\to\infty}\underline{S}_{P_k}(f)=A(f)$.

<u>Proof</u>

... Now for any two partitions P and P', the set $P\cup P'$ yields a partition Q that is the common refinement of P and P' ...[8]

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[5] Any map may be coloured by four colours. Proven with the assistance of a computer by Appel and Haken in 1976

[6] The Robbins conjecture is that any structure satisfying the equations: (1) $x+y=y+x$; (2) $(x+y)+z=x+(y+z)$; (3) $n(n(x+y)+n(x+n(y)))=x$ is a Boolean algebra.

[7] This is a statement made by Henson and quoted by Bringsjord and Arkoudas [2006] in Olszewski et al. [2006] p.72.

[8] This is an extract from p. 43 of Carter and van Brunt [2000] which is an introduction to the Lebesgue-Stieltjes Integral. I choose this example by randomly picking up a book on my desk, randomly opening it at any page and copying the first result I saw.

One does not need to understand this extract "mathematically" to see that it is *not* a first-order statement.[9] The expression *Riemann integrable* denotes a second order property of functions; a *partition* is a finite set of numbers, but we are *quantifying* over sequences of these, so over *sets of sets*, which are equivalent to a second-order property; we have a relation between partitions (first-order sets) of *refinement* – so that relation is second-order. In the proof we see that it progresses in natural language into which are interspersed various first-order symbols; I would like to know what the ontological status of a "common refinement" is. There is no prima-facie empirical evidence whatsoever to suggest that human beings *think in first-order set theory*. We work things out through meanings![10]

The prima facie alternative to formalism is the traditionally older philosophy that symbols are signs or tokens denoting *concepts*, which may also be called *meanings* or *intensions*. This philosophy claims that when we operate formally with symbols we do so on the basis of our understanding of those meanings – so it is the meanings that constrain the manipulations. It also claims that it is impossible to conceive of mathematics independently of the human endeavour to understand. Subsequently, I shall develop this into a neo-Kantian philosophy of mathematics. For the present I wish only to introduce it.

4 Conflation of the potential with the actual infinite

Among the concepts that are essential to our mathematical understanding are those of the potential and actual infinite. The distinction between these shall be vital to the development of this paper.

4.1 The potential and actual infinite

In mathematical discourse we meet two differing conceptions of the infinite: the potential infinite and the actual. The potential infinite is illustrated by the inexhaustibility of counting; for no mater how large a number we have reached it is always possible to count to a higher one merely by adding one more. In the actual infinite we conceptualise the entire process of counting as a *completed totality*.

[9] Second order statements quantify over properties, first-order statements do not.

[10] I am certainly not alone in making this assertion. For example, in Mayberry [1994], we have a critique of the "curious doctrine" that "mathematical logic is to be *identified* with first-order logic". Mayberry claims that the theories of topological spaces, Hilbert spaces, Banach spaces, Noetherian rings, cyclic groups (etc), are second-order theories. The eliminatory theories of arithmetic, geometry and analysis are also second-order theories. He remarks, "First-order logic is very weak, but therein, paradoxically, lies its strength. Its principal technical tools – the Compactness, Completeness, and Löwenheim-Skolem theorems – can be established only because first order logic is too weak to axiomatize either arithmetic or analysis." (p.411) He outlines the properties of second order logic, which is, nonetheless, "a powerful tool of definition: by means of it, and by means of it alone, we can capture mathematical structure up to isomorphism using simple axiom systems." (p.412)

> The potential infinite is symbolised by ∞, which represents the inexhaustibility of counting. The natural numbers are 0, 1, 2, … . Here the dots also indicate the potential infinite. When we wish to talk of all the natural numbers – and conceive of them as a collection[11] – we represent this by $\mathbb{N}$. It is important to bear in mind that this collection is a *potential infinity*.
>
> In the actual infinite we have a conception of an infinite collection of objects – usually points or numbers – *as given actually in its entirety*. The theory of actually infinite, or complete collections is *set theory*, which employs many symbols for such objects. The first infinite collection is denoted by ω. This is a collection of *finite ordinal numbers* – that is, numbers ordered in succession. Whereas the collection of natural numbers, $\mathbb{N}$, is not bounded above, the collection of all finite ordinals is bounded above. The ordinal ω is conceptualised as *another ordinal* following in succession after all the finite ordinals.

The two conceptions, $\mathbb{N}$ and ω, can be seen to be wholly different as ideas. Therefore, it would seem to be a fundamental error to conflate or equate the two, though this is in fact common practice.

> Ordinals are the order types of well-ordered sets. They are the infinite analogues of the natural numbers, and in many respects they behave like the latter ones. In fact, <u>the finite ordinals are the natural numbers</u>, and hence the transfinite class of ordinals can be considered <u>as an endless continuation</u> of the sequence of natural numbers. (Komjáth and Totik [2000] p. 37)

My underlining. The two concepts are so different *as concepts* that the ideas expressed in this quotation would seem to be simply erroneous. If it is an error, it is a common one. However, there is the usual formalist defence: -

> It is a philosophical quibble whether the elements of ω are the *real* natural numbers (whatever that means). The important thing is that they satisfy the Peano Postulates. (Kunen [1980] p.19)

The Peano Postulates (Chap. 2 /2.12) are a set of rules said to capture all the properties of the natural numbers. The formalist defence is that it is only the rules and the syntactic manipulations they satisfy that matter. Kunen rejects any attempt to identify what natural numbers might *really* be, which is consistent with his nominalism. It is a fundamental contention of this paper that the potential and actual infinite must not be conflated and that

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[11] I am implicitly employing a distinction here between collections that are sets and collections that may be something else [See Chap. 2, Sec 1.3].

the real distinction between them has also real effects both in terms of mathematical structures and the formal rules to describe those structures. This is immediately evident, for if by Peano Arithmetic we include the full second-order Principle of Complete Induction (Chap.2 / 2.12), then it is contentious to say that set theory based on ω satisfies the Peano Postulates, and a real formal distinction between the two theories exists.

5 The fundamental category

The *inexhaustibility* of the potential infinite arises in us from the iteration of counting (or, indeed, any process). We begin at 1 and by adding 1 more unit to it arrive at 2; we repeat the process and arrive at 3; and so on *ad infinitum*. The following is a seminal comment by Weyl: -

> My investigations began with an examination of Zermelo's axioms for set theory, which constitute an exact and complete formulation of the foundations of the Dedekind-Cantor theory. ... My attempt to formulate these principles as axioms of set formation and to express the requirement that sets be formed only by finitely many applications of the principles of construction embodied in the axioms – and, indeed, to do this *without presupposing the concept of natural* numbers – drove me to a vast and ever more complicated formulation but, unfortunately, not to any satisfactory result. Only when I had achieved certain general philosophical insights (which, incidentally, required that I renounce conventionalism[12]), did I realize that I was wrestling with a scholastic pseudo-problem. And I became firmly convinced (in agreement with Poincaré, whose philosophical position I share in so few other respects) that *the idea of iteration, i.e. of the sequence of natural numbers, is an ultimate foundation of mathematical thought* – in spite of Dedekind's "theory of chains" which seeks to give a logical foundation for definition and inference by complete induction without employing our intuition of the natural numbers. For if it is true that the basic concepts of set theory can be grasped only through this "pure" intuition, it is unnecessary and deceptive to turn around then and offer a set-theoretic foundation for the concept "natural number." Moreover, I must find the theory of chains guilty of a *circulus vitiosus.*[13] If we are to use our principles to erect a mathematical theory, we need a foundation – i.e. a basic category and a fundamental relation. As I see it, mathematics owes its greatness precisely to the fact that in nearly all its theorems what is essentially *infinite* is given a finite resolution. But this "infinitude" of the mathematical problems springs from the very foundation of mathematics – namely, *the infinite sequence of the natural numbers and the concept of existence relevant to it.* (Weyl [1994], p. 48 – 49)

[12] Conventionalism is an alternative name in this context for formalism.

[13] Vicious circle; i.e. guilty of circularity.

My underlining. Weyl correctly attributes to Poincaré the thesis that identifies the fundamental category of number as the foundation of arithmetic, together with its notion of potentially infinite iteration. Let us briefly sketch the philosophy of mathematics that acts as the alternative to formalism. This is the claim (1) that the sequence of natural numbers acts as a foundation of mathematical thought, it is a "basic category" and the idea of iteration is its "fundamental relation" and (2) that the principles of reasoning founded upon this fundamental relation, specifically, complete induction, is the basis of a *synthetic logic* that is not formalised in the analytic logic (first order mathematical logic).

Frege attempted to replace number as a category by sets [Chap.2, Sec.2.7.2]; it is agreed his attempt failed. It is agreed that any alternative definition of natural number from within set theory in terms of particular sets has failed. This is known as Benacerraf's problem. [Chap.16, Sec.3] If there are categories, then number is unchallenged as one of them. Nonetheless, formalists replace number by effective processes. So the challenge is to show concretely that effective processes cannot replace number in this respect. To do this requires a proof of Poincaré's thesis: that mathematical induction is a synthetic principle of reasoning.

One might object that there is nothing easier for a computer to do than count up. Adding 1 is surely the simplest recursive function [Chap2, Sec. 2.3] that there is, and is routinely proven to be so. The reply to this is that counting for us embraces the potential infinite – we not only grasp the potential infinite, we in some sense, attain it – whereas for a machine counting is just a process that generates a list. We program a computer to count up, and it starts to output the following: -

1 11 111 1111 11111 ...

The problem is with the dots. How does the computer "know" when to stop? If it does not stop then *can the machine be said to be counting?* It is producing an infinite list, but then it doesn't know when to stop and output something like, "This is an infinite list generated by the rule, add 1". If it does stop, *what tells it to stop?* Perhaps it has been programmed with a finite number whose role is to act as a cut off point – for example, stop if the number of digits = 100 and print, "This is an infinite list generated by the rule, add 1". In the first case the machine does not halt, and in the second case the machine halts because it is processing an analytic function on $n = 100$. It is computing a partition of 100 into 100 separate digits. Also, if it does stop it must be programmed to interpret some sequence of 1s to represent infinity; this is because everything infinite in meaning can only correspond to a finite sequence in a computer. Concerning the infinite computers run simulations upon finite strings. So there is every reason to conclude *prima facie* that a computer is by no means able to count in the sense that we can count. At best, computers may be said to analyse numbers, but they do not generate them.

ounting up *adds something new* each time 1 is joined to a previous total. In terms of a spatial analogy we are always progressing away from 1 and *expanding* our space[14]. The natural numbers are *synthetic on the way up and analytic on the way down.* Most particularly, the synthetic mode of generation contains a concept that no amount of analysis of any individual member of a sequence will reveal: *the potential infinite.*

6 Poincaré's thesis and proof by complete induction

Proof by complete induction and its equivalent method of infinite descent rest upon "*the idea of iteration, i.e. of the sequence of natural numbers*", which is "*an ultimate foundation of mathematical thought*". As Poincaré explains:

> The process is proof by recurrence. We first show that a theorem is true for $n=1$; we then show that if it is true for $n-1$ it is true for *n*, and we conclude that it is true for all integers. (Poincaré [1982], p.398)

This method of proof is represented schematically by: -

$$\frac{\begin{array}{l}P(1)\\ P(n)\rightarrow P(n+1)\end{array}}{(\forall n)P(n)}$$

The first line in this argument is known as the *particular step*, and the second line as the *induction step*. Both these steps encode finite information. The conclusion is one about a potentially infinite collection.[15] Mathematical induction has been likened to climbing up a

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[14] We learn to associate counting up at some very early age with space – lining up our building blocks in a row, or constructing them into a tower. But it is arguable that the structure created by counting is not a space at all, at least, not in the sense in which visible space is space.

[15] Example of a complete induction: Prove that n^3+5n is divisible by 6 for all *n*.

Proof: Let us use the symbol | to mean "divides into". For example 6 | 36 is read "6 divides into 36" or "36 is divisible by 6". (This is standard notation in number theory.) For the particular result, when *n* = 1

$n^3+5n=1+6=6 \quad \Rightarrow \quad 6\,|\,6$

so the result is true for *n* = 1. For the induction step assume that the result is true for *n* = *k*. That is $6\,|\,(k^3+5k)$. We have to show $6\,|\,\left((k+1)^3+5(k+1)\right)$. Now

$(k+1)^3+5(k+1)=k^3+3k^2+3k+1+5k+5=k^3+5k+3k^2+3k+6=(k^3+5k)+3(k^2+k+6)$

If *k* is an odd number then k^2+k is even (as the sum of two odd numbers), then k^2+k+6 is even and $6|3(k^2+k+6)$. If *k* is an even number then k^2+k+6 is even, and $6|3(k^2+k+6)$. In either case $6|3(k^2+k+6)$. By the induction hypothesis $6|(k^3+5k)$ so $6|\{(k^3+5k)+3(k^2+k+6)\}$, since 6 divides both ladder. The steps of the ladder correspond to numbers. The climb up the ladder would not be possible if the infinite chain of natural numbers was not already generated in our minds by the idea of repeated iteration of counting up one. This creates the potential infinity of all the natural numbers, and the argument by complete induction attaches to this chain a property of numbers that is inductively carried up the ladder of all the natural numbers.

This paper will illustrate how formal analytic logic is the science of inference derived from analogy with the relation of part to whole in space. Formal analytic logic is also sometimes called *syllogistic reasoning*. There is no prima facie reason to suppose that the "force" that constrains inference in the case of complete induction is represented by any formal analytic logic.

6.1 Poincaré's thesis

> In his essay *Mathematics and Logic*, Poincaré states that the principle of complete induction "appeared to me at once necessary to the mathematician and irreducible to logic." (Poincaré [1996] p. 148)[16] This claim shall be called in this paper *Poincaré's thesis*.

I shall defend this Poincaré's thesis. About the attempt to reduce all of mathematics to a species of analytic reasoning Poincaré wrote: -

> … syllogistic reasoning remains incapable of adding anything to the data given in it; these data reduce themselves to a few given axioms, and we should find nothing else in the conclusions.
>
> No theorem could be new if no new axioms intervened in its demonstration; reasoning could give us only the immediately evident verities borrowed from direct intuition; it would be only an intermediary parasite, and therefore should we not have good reason to ask whether the whole syllogistic apparatus did not serve to disguise our borrowing? …
>
> If we refuse to admit these consequences, it must be conceded that mathematical reasoning has of itself a sort of creative virtue and consequently differs from the syllogism.
>
> The difference must even be profound. We shall not, for example, find the key to the mystery in the frequent use of that rule according to which one and the same uniform operation applied to two equal numbers will give identical results.

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halves separate. Thus the result is true for $k+1$ and the induction step holds. Hence, by mathematical induction n^3+5n is divisible by 6 for all n

[16] The original publication was 1914.

All these modes of reasoning, whether or not they be reducible to the syllogism properly so called, retain the analytic character, and just because of that are powerless." (Quoted in Detlefsen [1990], p.291 of Jacquette [2002])[17]

7 Poincaré's petitio pincipii

Both Weyl and Poincaré believe that the proponents of set theory argue in a circle in claiming to define number from within set theory. Weyl calls it a *circulus vitiosus* and Poincaré a *petitio pincipii*.[18] Weyl's argument is based on the principle that number is the only possible foundation for arithmetic, and one argues in a circle if one attempts to define number, because it has already been assumed. In addition to offering this argument, Poincaré also claims that we need complete induction to justify the definitions of set theory, so one argues in a circle to then use set theory to define complete induction.

For example, one of Poincaré's counter-arguments to the thesis that induction is true by definition is that a definition is only valid if it is not self-contradictory. We usually show this by supplying what he calls an "example", that is, a model.

> But such a direct demonstration by example is not always possible. Then, in order to establish that the postulates do not involve contradiction, we must picture all the propositions that can be deduced from these postulates considered as premises, and show that among these propositions there are no two of which one is the contradiction of the other. If the number of these propositions is finite, a direct verification is possible; but this is a case that is not frequent, and moreover, of little interest.
>
> If the number of propositions is infinite, we can no longer make this direct verification. We must then have recourse to processes of demonstration, in which we shall generally be forced to invoke that very principle of complete induction that we are attempting to verify." (Poincaré [1996] p.153)

It is specifically with regard to number that he accuses the "logiticians"[19] of making a *petitio principii* – albeit "most skilfully concealed". He is sarcastic about Peano's symbolic logic. "This invention of Peano was first called *pasigraphy*; that is to say, the art of writing a treatise on mathematics without using a single word of ordinary language." (Poincaré [1996] p.156)

[17] The original is in Poincaré, *The Value of Science* (1905) in *The Foundations of Science*, ed., and trans. G. Halsted, The Science Press, 1946

[18] The term *petito principii* derives from Aristotle. Both terms mean *error by arguing in a circle* – assuming what one has to prove.

[19] Although Poincaré is habitually sarcastic, this is not a pejorative term but a contemporary nomenclature for the proponents of the new logic: Hilbert, Russell and Peano among others.

Referring to Peano's definition of $1 = \iota T'\{ko_{-}(u,h)t(u \in \text{One})\}$ he comments, "I do not understand Peanian well enough to venture to risk a criticism, but I am very much afraid that this definition contains a *petitio pincipii*, seeing that I notice the figure 1 in the first half and the word One in the second..." (Poincaré [1996] p.158) "What is zero? It is the number of elements in the class nil. And what is the class nil? It is the class which contains none." (Poincaré [1996] p.158) He calls such definitions "an abuse of the wealth of language". He sarcastically allows that Couturat's definition of 1 is more "satisfactory" because it uses the word two! "But I am afraid that if we asked M. Couturat what two is, he would be obliged to use the word one." (Poincaré [1996] p.159)

The founders of mathematical logic, which include Frege, Russell and Hilbert, probably did believe that they were breaking out of the fundamental category of natural number and reducing it to something analytic. [Defined Chap.2, Sec.2.1] Poincaré's critique arguably does successfully call them to account and challenge their assumptions. Strangely, as a critique of the modern formalism represented by the quotation from Curry above [Section 1 above] Poincaré's objections are quite ineffectual!

The reason for this is that whereas Frege, Russell and Hilbert were attempting to *justify mathematics* in the traditional sense of providing an epistemology for it, modern formalism attempts no such thing. It is the very essence of this formalism to deny all conceptual meaning to mathematical symbols and to regard whatever meaning they have as entirely reducible to the syntactic operations we perform with them. Their contention also is that every such operation can be reduced to a mechanical procedure such as could be effected by a computer. If formalists justify anything then it is by reference to more formal procedures – so one might say that *circularity is the very essence of their theory*. This is a feature that I shall also refer to as the *chicken and egg problem*.

7.1 The chicken and egg problem

This is illustrated as follows: -

1. Set theory is a first order theory.
2. Any model of a first-order theory is a set.

Similarly: -

1. There is a Boolean valued model of set theory.
2. Under the Stone representation theorem every Boolean algebra is isomorphic to a field of sets.

Any circularity of theories is defined to be a *chicken and egg problem*.

Modern formalism seems to have us running around in circles looking for a fundamental category. One wants to cry out: where do we start? What really is the origin of our ideas? For formalists all mathematical activity just concerns manipulations of forms – establishing correspondences between classes of formal systems. We cannot break out of a formal system, nor should we wish to. That is the theory.

8 Materialism, empiricism and the ethical significance of this question

As many have strong commitment to the ideologies of materialism and empiricism, it is appropriate to reflect briefly on the question: does Poincaré's thesis constitute a refutation of either of these? Being by instinct neither a materialist nor empiricist, I was surprised by my own conclusion – that the answer to this question was no; in fact, I even found myself moving more towards empiricism than I had ever thought possible. In rejecting strong AI and formalism one not is forced to reject either of these philosophies. For example, does any one seriously suggest that organic chemistry is a first-order theory? If at some large-scale level brain circuitry bears some resemblance to Boolean switches that by no means obligates one to adopt the position of strong AI. My impression concerning the development of "advanced physics" is that more and more physicists are recognising an element in matter that *might* even be described as *spiritual in aspect.* I cannot say what the prospects for monism are, but suggest that, in a 1,000 years time, or 10,000 years time, our notion of matter may have advanced so far as for us to be able to say, with some confidence, *mind is matter*; but in doing so, I believe that our conception of matter will acquire attributes that hitherto we thought of as spiritual. I have no wish to make a prediction. I wish merely to state my opinion that materialism does not stand or fall with strong AI.

But it is no mere digression if at this juncture I stress the *ethical significance of the question before us.* The belief in strong AI really has entered our popular consciousness, and it is no exaggeration to say that we are collectively engaged at this time, culturally speaking, in a radical transformation of our concept of *what it is to be human*, and, without wishing to elaborate, there are many manifestations of this change that some, including myself, find disturbing. The thesis of strong AI *should be tested to destruction.* The whole tenor of the debate up to this time has been *so many accumulations of victories for AI.* Does anyone report on the failures? Are we reminded in the Daily Press that Day of Turing's Prediction [Chap.13 Sec. 1] has long come and gone? Are we updated regularly on the failure of the Japanese Fifth Generation project?[20] Let us hope that our culture is not about to enter upon a thousand year cycle wedded to Illusion. The Ancient Egyptians built pyramids whose purpose was to make their Pharaohs immortal. No one today believes in their science, but their culture endured in *spiritual stagnation attended by human oppression for three thousand years.* The time has come to test to destruction the thesis that man is a digital computer.

[20] The project initiated by the Japanese government in 1982 to create a supercomputer as a basis for artificial intelligence. It may be claimed that that the project has failed.

CHAPTER 2

Preliminaries and pre-requisites

Preliminaries

The annotation convention (1.1) should be noted. In the *Preliminaries* section I discuss (1.2) The Church-Turing thesis and (1.3) Basic conceptions underlying set theory – here the distinctions between collections and sets, and between sets and classes are theoretically significant.

Prerequisites

This is an attempt merely to identify or list concepts or results from mathematics in general that underpin the discussion in this text. This section can be omitted in its entirety. Some cross-references in the text refer to these statements. The list is not exhaustive.

1 Preliminaries

1.1 Annotation (+)

Any result that I believe originates with this paper is marked by (+).

1.2 The Church-Turing thesis

There are several distinct mathematical descriptions of effective computation: -

1. Turing's analysis based on Turing machines.
2. The Gödel-Herbrand analysis based on recursive functions.
3. An analysis in terms of Abacus machines.
4. Church's analysis based on his lambda calculus.
5. Markov algorithms.
6. Post systems.

Here are some versions of the Church-Turing thesis that may be found in the literature: -

1. "Church's thesis: all computable functions are Turing computable." (Boolos and Jeffrey [1980] p. 54.)

2. "In around 1936, several mathematicians (Alonzo Church, Stephen Kleene, Emil Post, and Alan Turing) independently proposed precise definitions of the notion of effective procedure. Even though their definitions were very different from each other conceptually, it was proved that these definitions are all equivalent, in the sense that every function that can be computed under one definition can be computed using the others as well." (Wolf [2005] p.96)
3. CTT: "… no human computer, or machine that mimics a human computer, can out-compute a universal Turing machine" (p.10 of Abramson in Olsweski et al. [2006] who is quoting from Jack Copeland, 2002a, p.67)

Of these the last, CTT, seems to be taken as the basis of discussion in the anthology of articles entitled *Church's Thesis After 70 Years*, (Olszewski et al. [2006]). Nonetheless, in this paper the first two statements shall be taken to represent the Church-Turing thesis.

<u>Definition, The Church-Turing Thesis</u>
The *Church-Turing thesis* is the claim: -
1. The six analyses of what is effectively computable given above are all formally equivalent. Call this equivalence class the *class of all Turing computable functions*.
2. All effectively computable functions belong to the class of all Turing computable functions and conversely.

This definition equates a mathematically precise notion of *Turing computable function* to the intuitive notion of *effectively computable function*. (This point has been frequently expressed in the literature.) This gives the thesis a very peculiar status. Part (1) is a mathematical theorem that has been formally proven, and is thereby not contentious. Part (2) is said to be an empirical thesis and to be supported by "evidence" of an observational nature rather than mathematical.

> The equivalence of these definitions, as well as our substantial experience with computer languages and programs, provide strong empirical evidence that these definitions do in fact correctly represent the intuitive notion of an effective procedure, or mechanical computation. There is no way to prove this, but it is a standard view as a sort of informal axiom, called *Church's thesis.* (Wolf [2005] p.96)

It is a consequence of this empirical thesis that it is expected that should anyone come up with a new analysis of what an effectively computable function is, then it will be proven to belong to the class of all Turing computable functions. In the Church-Turing thesis an extension (the class of all Turing computable functions) is equated with an intension (all effectively computable functions). [See 1.3 below] Notwithstanding the empirical character of part 2 of the definition above, in this paper I shall take it as a definition of what *effectively computable* means. To demonstrate the pertinence of this, consider the following two quotations from the seminal paper *Computing Machinery and Intelligence* by Turing (Turing [1950]): -

1. ... we only permit digital computers to take part in our game. [p.7]
2. The idea behind digital computers may be explained by saying that these machines are intended to carry out any operations which could be done by a human computer. The human computer is supposed to be following fixed rules; he has no authority to deviate from them in any detail. We may suppose that these rules are supplied in a book, which is altered whenever he is put on to a new job. He has also an unlimited supply of paper on which he does his calculations. He may also do his multiplications and additions on a "desk machine", but this is not important. [p.8]

The underlining is my own. This is the clearest description of what is meant by "effective computation" and *following fixed rules without deviation* lies at the core of this. But even this is ambiguous, and the implicit notion of what we shall take for a rule serves to convert the discussion into a species of definition.

Turing also places emphasis on digital computers. If we are restricting our attention to digital computers we have certainly very good evidence of the mathematical variety for accepting the Church-Turing thesis. This concerns the structure of a digital machine that is made solely of binary switches. The binary switches restrict the digital computer theoretically to a very specific domain, whose topology can be precisely delimitated, at least in the sense of being given an upper bound [Chap. 7, Sec. 3] . Hence, here I adopt the Church-Turing thesis as a *definition* of what is effectively computable, and as a theorem about digital computers and reject the claim that it is an empirical thesis.

Church –Turing Theorem

The six analyses (1) Turing's analysis based on Turing machines, (2) The Gödel-Herbrand analysis based on recursive functions, (3) Abacus machines, (4) Lambda calculus, (5) Markov algorithms, (6) Post systems are all formally equivalent. Call this equivalence class the *class of all Turing computable functions.*

Definition, Church-Turing thesis, V2

The *Church-Turing thesis V2* is the statement that any *effectively computable function* is defined to be a Turing computable function.

The consequence of this is that the thesis is neutral as to the validity of strong AI. By defining effectively computable in terms of the equivalence class of Turing computable functions, it makes no direct claim as to whether *all mathematical proofs are Turing computable*: simply accepting this version Church-Turing thesis does not force one to accept strong AI. Nor does it provide any empirical evidence for strong AI. I do not adopt Copeland's version of the Church-Turing thesis, which is probably a disguised statement of the thesis of strong AI.

1.3 About basic set theory

1.3.1 Collections and sets

Formal logic is based on extensions not intensions. The term *intension*, which is a synonym of the term *concept*, denotes a mentally given *something* presented to consciousness. Another synonym of *intension* is *meaning.* A *collection* is a general, primitive notion for any gathering together of "things" into a single "whole" ["totality", "totum"]. It is ambiguous as to whether this act of collecting together is a mental act or is just given. A *fusion* is a collection of physical parts and is the sum total of its parts. A *multiplicity* is a collection that comprises a whole that is distinct from its *members*. Multiplicities may be *definite* or *indefinite*. In an *indefinite* (or *indeterminate*) multiplicity the members cannot be listed or determinately given by a rule; in a *definite multiplicity*, the members of the collection may be listed or determined by a rule. Multiplicities may be *extensional* or *intensional*: a multiplicity is extensional if it is determined solely by its members, so that two multiplicities with the same members must be identical; a multiplicity is intensional if it is determined by the concept that gives rise to the rule of membership, so that two multiplicities with the same members may yet be distinct if they are defined by different rules.

1.3.2 Definition, set

A *set* is a multiplicity that is definite and extensional.

Example

The null set is the definite, extensional multiplicity that has no members.

The null set is distinguished from a fusion, because there cannot be an empty fusion. The extension of the concept "unicorns in my garden" is the null set; the extension of the concept, "set that has no members" is the null set. These are different as intensions but identical as extensions. In set theory the two intensions are not distinguished. In set theory the *primitive notions* are *set* and *membership*. A set is a collection that has members. Members of sets are either sets or individuals. In *pure set theory* there are no individuals. Set theory rules out definition by abstraction: -

Definition by abstraction

Let φ denote an *intension* (*concept*, *meaning*). Then $\{x : \varphi(x)\}$ *defines by abstraction* a collection that satisfies a property that the intension φ denotes. We abstract from the intension (concept) to the set of entities that satisfies this property.

The *axiom of extensionality* explicitly disallows definition by abstraction from intensions. Set theory permits definition by formula in the form of the Axiom of separation.

1.3.3 Assumed knowledge of set theory

It is assumed that the reader is familiar with the basic operations of set theory: unions, intersections, complements, power set. Also equivalence, isomorphism, homomorphism.

1.3.4 Zermelo-Fraenkel set theory at a glance

Zermelo-Fraenkel set theory requires eight axioms: -

1. Axiom of extensionality
   Sets are identical if they have the same members
2. Axiom of regularity – also called the axiom of foundation
   A set cannot be a member of itself
3. Axiom schema of separation
   A property is collectivising if its members are already members of a set
4. Pairing axiom
   A pair of sets is another set
5. Union axiom
   The union of two sets is another set
6. Power set axiom
   Sets have subsets and the set of all subsets is another set
7. The axiom schema of replacement, sometimes also called the sum axiom
   The images of functions are also sets
8. Axiom of infinity
   There is at least one collection with an actual infinite number of members

1.3.5 **Proper classes, classes and sets**

Russell's paradox [Proven in Chap. 16.4.5], demonstrates that there can be no such thing as a universal set containing all individuals [See below Sec. 2.1] . It shows that the universal set is a self-contradictory concept. However, it is customary in set theory to allow discussion of a universal *collection*. This is in accordance with the doctrine of "limitation of size" – a set is something that is "not too big" in the sense of definitely constructed "from below" by certain operations or rules. The problem with the universal *collection* is that it is too big to be a set. Hence, the term "proper class" is reserved to designate those collections that are too big to be sets. The term "class" is used to designate any collection that is either a proper class or a set.

1.3.6 Definition, proper class

A *proper class* is a collection that has members that are sets but cannot be constructed by the rules or operations of set theory.

2 Prerequisites

2.1 Epistemology

A posteriori: knowledge that is gained from particular experience.

A priori: knowledge that cannot be derived from particular experience.

Subject / Predicate: In a statement the subject is what the statement is about and the predicate is what is said of the subject.

Analytic: the meaning of the predicate is contained in the meaning of the subject.
Synthetic: the meaning of the predicate adds new meaning to that of the subject.
Analytic a priori: true by definition or convention; the meaning of the predicate is contained in the meaning of the subject.
Synthetic a posteriori: substantive knowledge of the world derived from particular experience.
Synthetic a priori: substantive knowledge of the world, neither true by definition nor derived from particular experience.
Individual: whatever can be the referent of the logical subject of a statement.
Contingent: a statement whose truth depends on some particular state of affairs in the world.
Necessary: a statement that is true in all possible worlds.

2.2 Logic

It is assumed that the reader has some knowledge of formal logic – the propositional and predicate calculus. In some respects this text does not assume much more knowledge than the following: -

> Truth functions, truth tables, tautology, rules of inference, rule of assumptions, conditional proof, deduction theorem, quantifiers, free and bound variables, generalisation, logical truth, recursive definition of a well formed formula (abbreivated by wff.), prenex normal form.

2.2 Turing machines

A Turing machine is a device for performing a computation. It may be visualised as a car moving along a *track* or *tape* that is divided into segments and is potentially infinite in length. The segments contain *symbols*. A result shows that only two different symbols are required. These are designated 0 and 1. The car scans one segment of the track at a time. The car has a *program* that instructs it what to do when it scans a symbol on the track. The instruction depends on (a) the *state* the car is in when it scans the symbol and (b) on the symbol. These two pieces of information are sufficient to instruct the car to perform an *action* and to tell it which state to go to next. This information is encoded in a *quadruple*, so the mechanical description of the machine is an implementation only of an abstract structure.

2.2.1 Definition, quadruple

A *quadruple* is an ordered 4-tuple of the form $\left(q_i\ \sigma_i\ \sigma_i'\ q_i'\right)$ where q_i, q_i' are states, $\sigma_i \in \{0,1\}$ and σ_i' is an instruction, $\sigma_i' \in \{0,1,L,R\}$.

2.2.2 Proposition (Boolos and Jeffrey [1980] p. 30)

Any function from positive integers to positive integers which is Turing computable is Turing computable in monadic notation by a Turing machine which uses only the symbols 0 and 1.

2.2.3 Definition, action

An action is one of the following:-

$0:1$ Change the scanned symbol on the tape to 1. Similarly, for $1:0$.

$0:L$ On scanning the symbol 0 move left. Likewise, $1:L$.

$0:R$ On scanning the symbol 0 move right. Likewise, $1:R$.

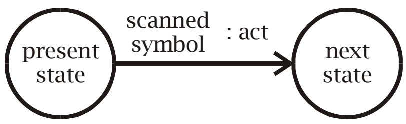
Example

$(2\ 1\ L\ 5)$ is read "When in state 2 scanning 1 move left and enter state 5".

2.2.4 Definition, program, Turing machine

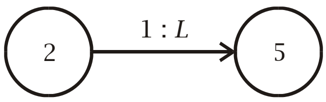
A *program* is a set of quadruples. A program is also called a *Turing machine.*

It is useful to have a diagrammatic representation of a Turing machine. The *flow graph* introduced by Boolos and Jeffreyn [1980] is very helpful: -



Example

The quadruple $(2\ 1\ L\ 5)$ has flow graph



2.2.5 Some nomenclature used in this text

In this text the tape is called the *Turing tape*. The tape has an infinite assignment of 0s and 1s. A *finite configuration* of the tape shall be an assignment of 0s and 1s to the tape that can be encoded in a finite series of instructions or rules. For example, the tape assignment: -

1 0 1 1 0 1 1 1

together with the rule that the tape has infinite 0s to the left and right of this *block*. The *standard configuration* is any bloc of *n* 1s on a tape that otherwise contains 0s. It is evident that it is a finite configuration.

2.3 Recursive functions

It is assumed that the reader is familiar with the concepts of *primitive recursion* and definition by *minimization*: " the smallest y for which $f(x_1, \ldots, x_n, y) = 0$ ". The following example of a *generating sequence* for the definition of addition by primitive recursion is given as an *aide memoir: -*

Example

Addition is PR. That is: -

$m+0=m \qquad m+(n+1)=(m+n)+1$

with generating sequence: -

1. $f_1(x)=x$ identity
2. $f_2=x+1$ successor
3. $f_3(x,y,z)=z$ projection
4. $f_4=f_2\circ f_3$ composition
5. f_5 defined by recursion with $h=f_1$, $g=f_4$; f_5 is addition.

To compute $2+3=5$ by this sequence: -

$$\begin{aligned} f_5(2,3)=2+3=f_4(2,2,2+2)=f_2\circ f_3(2,2,2+2)=f_2(2+2)=s(2+2)=s(2+f_4(2,2,2+1)) \\ =s(f_2\circ f_3(2,2,2+1))=s(f_2(2+1))=s(s(2+1))=s(s(f_4(2,2,2))) \\ =s(s(f_2\circ f_3(2,2,2)))=s(s(f_2(2)))=s(s(s(2)))=5 \end{aligned}$$

Recursive functions are functions that are primitive recursive with the addition of the operation of minimilization.

2.4 Arithmetic Hierarchy

2.4.1 Measure of complexity

Let a formula be in prenex normal form, ϕ (i.e all quantifiers are located at its beginning.) The number of *alternations* of quantifiers (changes between $\forall,\exists$) is a measure of the formula's complexity.

Σ_0,Π_0 ϕ is quantifier-free.

Σ_1 ϕ begins with an existential quantifier and has 0 alternations.

Π_1 ϕ begins with a universal quantifier and has 0 alternations.

Σ_{n+1} ϕ begins with an existential quantifier and has n alternations.

Π_{n+1} ϕ begins with a universal quantifier and has n alternations.

A finite string of quantifiers of the same type can be rewritten as a single quantifier.

Examples

Σ_1 $(\exists x)Px$

Π_1 $(\forall x)Px$

Σ_2 $(\exists x)(\forall y)(Px\vee Qy)$

$(\exists x)(\forall r)(\forall s)(\forall t)(P(x,r,s,t)\supset Qx\wedge S(r,s))$

Π_2 $(\forall u)(\forall v)(\exists x)(\exists y)\psi$ where ψ is a quantifier-free formula.

These categories are technically disjoint; that is, for example: -

$\Pi_n \cap \Pi_{n-1} = \varnothing$
$\Pi_n \cap \Sigma_n = \varnothing$

However, any formula of lower complexity can be translated into a formula of higher complexity by the addition of dummy variables; hence a hierarchy of embeddings can be defined.

Examples

$$\begin{array}{lcl} \Pi_1 & & \Sigma_2 \\ (\forall x)(\forall y)(xy = yx) & \rightarrow & (\exists z)(\forall x)(\forall y)(xy = yx) \end{array}$$

Likewise,

$$\begin{array}{lcl} \Pi_1 & & \Pi_2 \\ (\forall x)(\forall y)(xy = yx) & \rightarrow & (\forall x)(\forall y)(\exists z)(xy = yx) \end{array}$$

2.4,2 Arithmetic hierarchy

We have the arithmetic hierarchy

$$\begin{array}{ccccccccccccc} & & \Sigma_1^0 & & & & \Sigma_2^0 & \ldots & & & \Sigma_\alpha^0 & & & \ldots \\ & \nearrow & & \searrow & & \nearrow & & & & \nearrow & & \searrow & & \\ \Delta_1^0 & & & & \Delta_2^0 & & & \ldots & \Delta_\alpha^0 & & & & \Delta_{\alpha+1}^0 & \ldots \\ & \searrow & & \nearrow & & \searrow & & & & \searrow & & \nearrow & & \\ & & \Pi_1^0 & & & & \Pi_2^0 & \ldots & & & \Pi_\alpha^0 & & & \ldots \end{array}$$

where Δ_n denotes the *ambiguous class*:

$\Delta_{n+1} = \Sigma_n \cup \Pi_n$ either Σ_n or Π_n
$\Delta_n = \Sigma_n \cap \Pi_n$ both Σ_n and Π_n

2.4.3 Definable set

Σ_0, Π_0 Sets that are quantifier-free definable. Also said to be Σ_0 -definable and Π_0 -definable.

Σ_n, Π_n Sets that are definable by formulas of Σ_n, Π_n complexity respectively.

$\Delta_{n+1} = \Sigma_n \cup \Pi_n$ either Σ_n or Π_n definable
$\Delta_n = \Sigma_n \cap \Pi_n$ both Σ_n and Π_n definable

2.5 Order relations

2.5.1 Definition, ordered set and order relation

An ordered set is any set on which an *order relation* can be defined. An order relation is a relation $\le$ on a set X that satisfies the following four axioms

O1 Reflexive: For all $x \in X$, $x \le x$

O2 Transitive: For all $x, y, z \in X$, if $x \le y$ and $y \le z$ then $x \le z$

O3 Antisymmetric: For all $x, y \in X$, if $x \le y$ and $y \le x$ then $x = y$

O4 For all $x, y \in X$, either $x \le y$ or $y \le x$. In this case x, y are said to be *comparable*.

2.5.2 Definition, partial order, poset

If a structure $(X,\leq)$ satisfies O1, O2 and O3 then the relation $\leq$ is called a *partial order*, and the set *X* is called a *partially ordered set,* or *poset.*

2.5.3 Definition, total order, chain

If in addition to being a partial ordering $(X,\leq)$ satisfies O4, then the relation $\leq$ is called a *total order* or *linear ordering*, and the set *X* is called a *totally ordered set* or *chain.*

2.5.4 Definition, well-ordered

Let $(X,\leq)$ be a totally ordered set. Then *X* is said to be well ordered if and only if every non-empty subset *Y* of *X* contains a minimal element; that is, there exists an element $y \in Y$ such that for all $x \in X$, $y \leq x$. This element *y* is said to be the least element of *Y.*

Example, the set of natural numbers

The set $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is an ordered set, meaning, every element, after the initial element 0, has an immediate successor. The set $\mathbb{N} = \{0, 1, 2, 3, ...\}$ is a totally ordered set, where $\leq$ is the usual relation of 'greater than or equal' numbers. The set $\mathbb{N}$ with the standard ordering $\leq$ is well-ordered. Under this ordering every non-zero number $n \in \mathbb{N}$ has a unique predecessor.

2.6 Maximal elements

2.6.1 Definition, maximal, minimal element

Let *X* be a partially ordered set. Then an element $x \in X$ is said to be *maximal* if for all $y \in X$, $y \geq x \Rightarrow y = x$. This states that there is no element in *X* other than *x* itself that is greater than or equal to *x.* **A** minimal element of a subset *X* of a partly ordered set *P* is an element *a* such that $a < x$ for no $x \in X$. It is a result that a maximal element is unique if $(X,\leq)$ is a total order.

2.6 2 Definition, bounded above

Let *S* be a set of real numbers. If there is a real number *K* such that, for every member $x \in S$, $x \leq K$ then *S* is said to be *bounded above*, and *K* is called *an upper bound* of *S.* Likewise, if there exists a *k* such that, for every member $x \in S$, $x \geq k$, then *k* is called *a lower bound* of *S.*

2.6.3 Definition, Least upper bound, greatest lower bound

Any number that is greater than every element of a set *S* may serve as an upper bound for the set; but there may be just one number among these upper bounds that is *the least upper bound.* The least upper bound for a set *S* is a number *K* that is an upper bound for *S* such that, if ε is any small positive number, then $K - \varepsilon$ is greater than some member of the set *S.* Likewise, *the greatest lower bound* of *S* may be defined. The least upper bound is also called the *supremum*, denoted *sup*, and the greatest lower bound is also called the *infimum,* denoted *inf.*

2.7 Cardinal numbers

2.7.1 Hume's principle

Hume's principle is as follows: the number of *F's* is equal to the number of *G*'s iff there is a one-one correspondence between the *F*'s and the *G*'s. For example, at a dinner party, the number of knives will equal the number of forks if, and only if, for every knife there is a fork. We do not in fact need to know the number of knives (say "six") in order to establish this correspondence.

2.7.2 The Fregean concept of number

Frege used Hume's principle as the basis for his definition of number. Thus, for example, the number 2 becomes the name of the class of all sets of objects in one-one correspondence with any pair. The number 0 is the name of the empty set. (There is only one empty set.) In order to complete the definition of number Frege also needed to establish that the cardinal numbers so defined follow in a sequence: 0, 1, 2, 3, ... (So the Fregean definition of cardinal number given here is not his complete theory.)

2.7.3 Equinumerous sets

$X \sim Y$ iff there is a one-one mapping from X onto Y. X and Y are said to be *equinumerous* or *equipotent*. It is a result that $X \sim Y$ is an equivalence relation.

2.7.4 Cardinal number

To each set X we assign a cardinal number, denoted $\operatorname{card} X$.

$\operatorname{card} X = \operatorname{card} Y$ **iff** $X \sim Y$ $\qquad$ $\operatorname{card} X = 0$ **iff** $X = \varnothing$

The cardinal number of a finite set is the number of its elements.

$\operatorname{card} A < \operatorname{card} B$ **iff** $A \sim X$ where $X \subset B$

Cardinal numbers are equivalence classes under the relation of equinumerosity.

<u>Notation</u>

The notation $|X|$ is also used for $\operatorname{card} X$; that is, $|X| = \operatorname{card} X$.

2.7.5 Schröder-Bernstein theorem

$\operatorname{card} A \le \operatorname{card} B$ and $\operatorname{card} B \le \operatorname{card} A \Rightarrow A = B$

2.7.6 Cantor's theorem

The power set of any set X (finite or infinite) has cardinality strictly larger than that of X.

$\operatorname{card} X < \operatorname{card}(P(X))$

Equivalently: Let X be any set and $P(X)$ its power set. Then there does not exist a bijection φ such that $(\forall Y \in 2^X)(\exists x : \varphi(x) = Y)$

2.7.7 Theorem

Let $\operatorname{card} X = x$. Then $\operatorname{card}(P(X)) = \operatorname{card}(2^x) = 2^{\operatorname{card} X}$.

2.7.8 Cantor's anti-diagonalisation[1] theorem for the real numbers

The set of real numbers, $\mathbb{R}$, is not equipotent to the set of natural numbers, $\mathbb{N}$.

2.7.9 The continuum

The real number line is also called *the continuum.* It is a result that $\mathbb{R} \sim (0,1)$

2.7.10 Definition, power continuum

A set is said to be of *power continuum* if it is equipotent to $\mathbb{R}$.

Notation

c denotes the cardinality (power) of the continuum.

2.7.11 Definition, aleph (temporary)

$\text{card}\,\mathbb{N} = \aleph_0$. That is, $\aleph_0$ denotes the number of elements in a countably infinite set.

2.7.11 Results

1. $\aleph_0 + c = c$
2. The smallest transfinite cardinal number is $\aleph_0$.
3. $c = 2^{\aleph_0}$

2.8 Ordinal numbers

2.8.1 Natural numbers, ordinals and cardinals

Ordinal numbers are expressed in natural language using the terms, "first", "second", "third" and so forth. These contrast with natural counting numbers, "one", "two", "three" and so on. Ordinal numbers relate to position within an order, and natural numbers relate to the size of a collection. The answer to the question, "What was the runner's position in the race?" is an ordinal number; the answer to the question, "How many people entered the race?" is a natural number.

2.8.2 Definition, order-isomorphism, similar

Let $(X,\le)$ and $(Y,\le_*)$ be well-ordered sets. An *order-isomorphism* is a bijection $f: X \to Y$ such that $x_1, x_2 \in X$ and $x_1 \le x_2 \Rightarrow f(x_1) \le_* f(x_2)$. Then X and Y are said to be *order-isomorphic*, or *similar*, denoted: $(X,\le) \approx (Y,\le_*)$ or simply, $X \approx Y$.

2.8.3 Theorem

Order-isomorphism, $\approx$, is an equivalence relation.

2.8.4 Definition, isomorphism type, ordinal (temporary)

An isomorphism type of ordered sets is called an *order type*, (after Cantor 1895). The order types of well-ordered sets are also called *ordinals.* Under this provisional definition, each well-ordered set $(X,\le)$ is a member an *ordinal number*, denoted $\text{ord}(X,\le)$. The ordinal number of any two order-isomorphic sets is the same. That is: -

$\text{ord}(X,\le) = \text{ord}(Y,\le_*)$ iff $(X,\le) \approx (Y,\le_*)$

[1] Sometimes "diagonalisation" is used for this argument. But we use "anti-diagonalisation" to distinguish it from the preceding argument. Anti-diagonalisation shows that sets *can not* be paired off, whereas diagonalisation shows that they can.

2.8.5 Theorem, Burali-Forti

The class of all ordinals is a proper class.

2.8.6 Theorem

Any two finite sets that have the same number of elements are order-isomorphic.

Notation

1. $\operatorname{ord}(X,\leq)=0$ iff $X=\varnothing$
2. If $(X,\leq)$ is well-ordered such that X contains $n\in\mathbb{N}$ elements, then $\operatorname{ord}(X,\leq)=n$
3. We denote the order number of the set $\mathbb{N}$ by ω. That is $\omega=\operatorname{ord}(\mathbb{N},\leq)\approx\{0,1,2,....\}$

Whereas a given well-ordered set may have only one cardinal number, under different orderings the set may have distinct ordinal numbers.

2.9 Minimal topology

2.9.1 Assumed knowledge about vector spaces

For example; vectors, vector spaces, orthogonality, scalar product, independent vectors, vector basis, span.

2.9.2 Formal definition, topological space

Let X be a non-empty set. A class **T** of subsets of X is called a *topology on X* if it satisfies the following two conditions: -

(1) the union of every class of sets in **T** is a set in **T**

(2) the intersection of every finite class of sets in **T** is a set in **T**.

2.9.3 Definition, neighbourhood

The *neighbourhood* of a point or open set in a topological space is an open set containing the point or set.

2.9.4 Definition, open cover

Let X be a topological space and $\{A_i\}$ a set of open subsets of X. Then $\{A_i\}$ is said to be an *open cover* of X if $x\in X\Rightarrow x\in A_i$ for some i; that is $\bigcup_i A_i=X$. A *subcover* is a subset of an open cover that is also an open cover.

2.9.5 Definition, compact space

A *compact space* is a topological space in which every open cover has a finite subcover. A *compact subcover* is a subspace of a topological space that is also compact and is also a topological space.

2.9.6 Definition, locally compact

A topological space X is *locally compact* iff every point in X has a compact neighbourhood.

2.9.7 Definition, analytic basis

Let (X,T) be a topological space. A *topological basis* for T is a subcollection $B \subset T$ such that every set in T is a union of sets from B.

2.9.8 Definition, metric

Let X be a non-empty set. A *metric* on X is a real function $d : X \times X \to \mathbb{R}$ of ordered pairs of elements of X such that

(1) $d(x,y) \geq 0$ for all $x, y \in X$

(2) $d(x,y) = 0 \Rightarrow x = y$

(3) $d(x,y) = d(y,x)$ [symmetry]

(4) $d(x,y) \leq d(x,z) + d(z,y)$ [triangle inequality]

These conditions above are also known as the *Hausdorff postulates*.

2.9.10 Definition, metric space

A *metric space*, denoted (X,d), is a non-empty set X equipped with a metric d on X. The elements of X are called *points* of the metric space (X,d).

2.10 The completeness axiom

The following are equivalent statements of the Completeness Axiom

2.10.1 Bolzano-Weierstrass theorem,

Every infinite bounded subset has a limit point in the set. In its original formulation this was expressed as: Every bounded sequence in Euclidean space $\mathbb{R}^n$ has a convergent subsequence.

2.10.2 Cauchy convergence criterion,

Let S by a non-empty subset of $\mathbb{R}$. Every Cauchy sequence on S converges to a real point in S. [We do not need *Cauchy sequence* in this text, so leave it undefined here.]

2.10.3 Dedekind completeness axiom

Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound in the set.

2.10.4 Cantor's nested interval principle

Given any nested sequence of closed intervals in $\mathbb{R}$, $[a_1,b_1] \supseteq [a_2,b_2] \supseteq \ldots \supseteq [a_n,b_n] \supseteq \ldots$ there is at least one real number contained in all these intervals, $\bigcap_{n=1}^{\infty}[a_n,b_n] \neq \varnothing$ bound in the set.

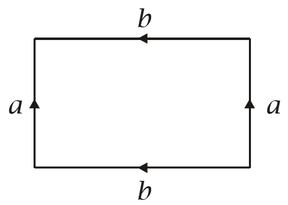
2.10.5 Heine-Borel theorem

Let X be a closed, bounded set on the real line $\mathbb{R}$. Then every collection of open subsets of $\mathbb{R}$ whose union contains X has a finite subclass whose union also contains X.

2.11 Algebraic topology

2.11.1 Edge equation

Example: identification of the edges of the sheet of paper by matching the symbols and arrows results in a torus.



2.11.2 Topological invariant

A property of a surface that is unaltered under continuous deformation.

2.11.3 Definition, Euler characteristic

Let Σ be a polyhedron. Let: -

V = number of vertices of Σ E = number of edges of Σ F = number of faces of Σ

Then, for any Σ the *Euler characteristic* is the number: $\chi = V - E + F$.

It is a result that the Euler characteristic is a topological invariant of a surface.

2.11.4 Canonical two-dimensional surfaces

Surface	Algebraic edge equation of quadrilateral	Other edge representations	Euler characteristic
Disk	$abcd = 1$	$a = 1$	1
Sphere	$abb^{-1}a^{-1} = 1$	$aa^{-1} = 1$	2
Torus	$aba^{-1}b^{-1} = 1$		0
Cylinder	$aba^{-1}c = 1$		0
Möbius band	$abac^{-1} = 1$	$dde = 1$	-1
Klein bottle	$abab^{-1} = 1$		0
Projective plane	$abab = 1$	$aa = 1$	1

2.12 Peano Postulates

The following axioms characterise arithmetic: -

P1 0 is a natural number

P2 If x is a natural number, there is another natural number denoted by x'. It is called the *successor* of x.

P3 $0 \neq x'$ for any natural number x.

P4 If $x' = y'$ then $x = y$

P5 Principle of Induction.
If Q is a property which may or may not hold of natural numbers, and if
(1) 0 has the property, and
(2) whenever a natural number x has the property Q, then x' has the property Q, then all natural numbers have the property Q.

The last of these statements is second-order – it quantifies over properties. There are continuum many properties, so the collection is not effectively computable. Arguably an effectively computable collection is provided by *first-order Peano Arithmetic*, in which P5 is replaced by an Axiom Schema: -

P5* For any wff $A(x)$: $A(0) \supset \big((\forall x)(A(x) \supset A(x')) \supset (\forall x)A(x)\big)$.

There are only a countably infinite number of such axioms, so the first-order Peano Arithmetic does not characterise Peano Arithmetic.

CHAPTER 4

Formal Analytic Logic

1 Poincaré's thesis

Analytic logic, which is also called *syllogistic logic*, is based upon analysis of the spatial relation of part to whole. The subject of arithmetic is $\mathbb{N}$, the collection of all natural numbers. This domain *is equipped with a principle of reasoning* called complete induction. I shall advance the thesis, originally proposed by Poincaré, that complete induction cannot be derived from analytic logic. Poincaré calls complete induction "reasoning by recurrence" and states that it is the creative principle in all mathematical reasoning that enables us to synthetically reduce the infinite to the finite. The following are quotations from his essay, *On the nature of mathematical reasoning*, where he states the thesis and advances important arguments in its favour: -

(*a*) ... mathematical reasoning has of itself a kind of creative virtue, and is therefore to be distinguished from the syllogism.

(*b*) A real proof, on the other hand, is fruitful, because the conclusion is in a sense more general than the premises.

(*c*) The essential characteristic of reasoning by recurrence is that it contains, condensed, so to speak, in a single formula, an infinite number of syllogisms.

(*d*) To prove even the smallest theorem he must use reasoning by recurrence, for that is the only instrument which enable us to pass from the finite to the infinite.

(*e*) Why then is this view imposed upon us with such an irresistible weight of evidence? It is because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it, and thereby of becoming conscious of it." (Poincaré [1982] pp 394 - 402]

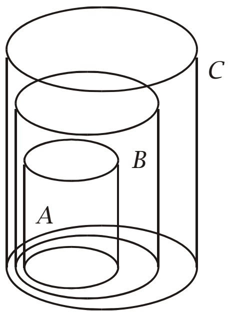
In order to evaluate this thesis we require a deep understanding of the nature of formal analytic logic.

2 Analytic logic and Boolean lattices

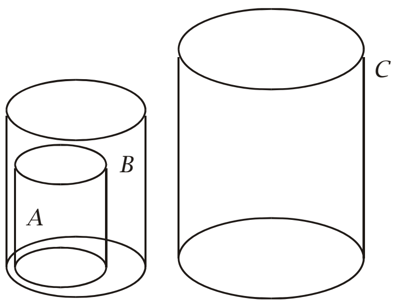
Modern formal logic is a rigorous expression of the syllogistic logic of Aristotle. Regarding the foundation of his logic, Aristotle wrote: -

> Whenever three terms are so related to one another that the last is contained in the middle as in a whole, and the middle is either contained in or excluded from the first as in or from a whole, the extremes must be connected by a perfect syllogism. By a middle term I mean one that is itself contained in another and contains another in itself; this term also becomes middle by position. By extremes I mean both that term which is itself contained in another and that in which another is contained. (Aristotle [1964 / c.400 BC] p. 8)

My underling. The extract illustrates how the idea of syllogistic logic arises from an *analogy* with containment in space, and is developed from the relation of part to whole.



If *A* is contained in *B* and *B* in *C* then *A* must be contained in *C*.



If *A* is contained in *B* and *B* is wholly separate from *C*, then the *A* *cannot* be contained in *C*. To say that *A* is contained in *C* is a *contradiction*. Every tautology is founded on a relation of containment in space. To illustrate this, consider the tautology, $\vdash (p \wedge q) \supset (p \vee q)$. This concerns two propositions, *p* and *q*, each of which can be negated to give $\neg p$ and $\neg q$. Let there be a partition of space to which these four propositions, $p, \neg p, q, \neg q$ apply. We cannot have $p \wedge \neg p$, for that is as much as to say that a partition is not contained in itself, which violates geometric intuition. Likewise, $q \wedge \neg q$ is impossible. All other combinations of $p, \neg p, q, \neg q$ are contingently possible. Hence, the space may be divided into four partitions: -

$\{1\}$ $p \wedge q$ $\{2\}$ $p \wedge \neg q$ $\{3\}$ $\neg p \wedge q$ $\{4\}$ $\neg p \wedge \neg q$

As indicated, let us identify these partitions with sets $\{1\},\{2\},\{3\},\{4\}$. Here the numerals 1, 2, 3, 4 signify nothing more than arbitrary distinct names for the undifferentiated content of each partition – whatever state of affairs it is in the world that makes the propositions true. The do not necessary stand for numbers. Let us call these four partitions *atoms* of the space generated by contingent propositions *p* and *q*.

$p \wedge \neg q$ $\{2\}$	$\neg p \wedge \neg q$ $\{4\}$
$p \wedge q$ $\{1\}$	$\neg p \wedge q$ $\{3\}$

The proposition *p* corresponds to the union of partitions $\{1\},\{2\}$, which is $\{1,2\}$ and $p \vee q$ corresponds to the partition $\{1,2,3\}$. Hence, since partition $\{1\}$ is contained in partition $\{1,2,3\}$, if $p \wedge q$ is true then $p \vee q$ must be true; this gives $(p \wedge q) \vdash (p \vee q)$ and the tautology $\vdash (p \wedge q) \supset (p \vee q)$ follows. There are 16 combinations of *p, q* corresponding to 16 partitions of the space represented by $\mathbf{1} = \{1,2,3,4\}$.

2.1 Geometric definition of a lattice

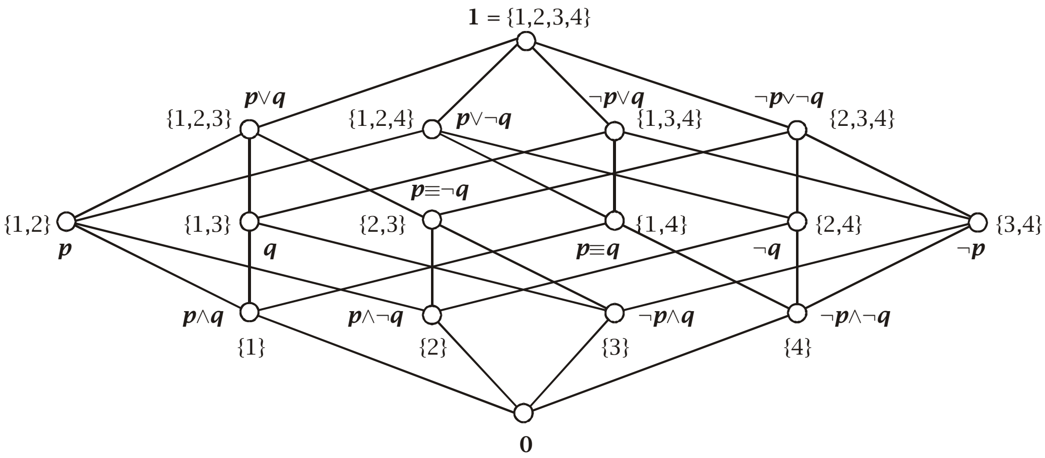
A poset [Chap. 2, Sec. 2.5.2] L is called a *lattice* if for every $x, y \in L$ $\sup(x,y) \in L$ and $\inf(x,y) \in L$. Let $x \vee y = \sup(x,y) \in L$ and $x \wedge y = \inf(x,y) \in L$. The element $x \vee y$ is called the "join" of *x* and *y* and the element $x \wedge y$ is called the "meet" of *x* and *y*.

2.2 Largest and smallest elements

The symbol **1** (bold typeface) denotes the largest element in the lattice and is the join of all the atoms. Likewise, the symbol **0** denotes the smallest element of the lattice, which is the meet of all the atoms.[1]

From this definition we can construct the *model* of the propositional logic of *p* and *q*, which is a Boolean lattice – that is, a complemented, distributive lattice.

[1] Not all lattices have a **1** and **0**. *Complete* lattices do have them. [9.4 above] All finite Boolean lattices are complete.



2.3 Boolean lattice of two propositions

This lattice shall be denoted, $\mathbf{2}^4$, where $\mathbf{2} \equiv \{0,1\}$ and

$$\mathbf{2}^4 \equiv \mathbf{2} \times \mathbf{2} \times \mathbf{2} \times \mathbf{2} \cong \{0,1\}^4 \cong \{0, p \wedge q\} \times \{0, p \wedge q'\} \times \{0, p' \wedge q\} \times \{0, p' \wedge q'\}$$

(Cartesian product) Here $p' = \mathbf{1} - p$ denotes the complement of *p*.

The properties of $\mathbf{2}^4$ are significant because they are inherited by all finite Boolean lattices.

2.4 Definition, Boolean lattice / algebra

A *Boolean lattice*, also called a *Boolean algebra*, is any structure $\langle B, \vee, \wedge, ', \mathbf{0}, \mathbf{1} \rangle$, subject to the axioms:

$(B1)$ $\langle B, \wedge, \vee \rangle$ is a distributive lattice[2]

$(B2)$ $p \wedge \mathbf{0} = \mathbf{0}$ $p \vee \mathbf{1} = \mathbf{1}$

$(B3)$ $p \wedge p' = \mathbf{0}$ $p \vee p' = \mathbf{1}$

I shall be using the terms *lattice* and *algebra* interchangeably, but it is useful to appreciate their different *nuances*. The term *lattice* emphasises the structural aspect – an abstract and rigid object that is visualised in the above diagram by the points and the lines joining them. The term *algebra* emphasises the relation to the language used to describe this structure. So we need both terms. This *lattice / algebra* is also related to *the logic built over the lattice*. Formal analytic logic is a structure built for the purpose of conducting inferences, whereas the lattice is a structure conceived as a collection of relations given *a priori*. Analytic logic is the application of the analytic properties of a lattice to the purpose of inference. It is customary to use the same symbols for the join $(\vee)$ and meet $(\wedge)$ in the lattice as those used for the logical operations of disjunction $(\vee)$ and conjunction $(\wedge)$. This is an abuse of notation

[2] A lattice is distributive iff the identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold in it.

nonetheless, because of the intimate relation between the logic and the lattice, it is natural and convenient. The lattice complement $(\,'\,)$ corresponds to negation $(\neg)$ in logic. The top and bottom elements, **1** and **0** are said to be "distinguished elements"; this just simply means that they are marked out in our language by distinct symbols; we do not use $p \vee p'$ for **1**; it would be inconvenient.[3] In the discussion that follows any general property of $\mathbf{2}^4$ is inherited by all finite Boolean algebras.

Concerning Boolean algebras, there is an important principle of duality: -

2.5 Principle of duality

Every Boolean algebra has an isomorphic dual formed by interchanging **0** for **1** and $\vee$ for $\wedge$ at every lattice point.

3 Boolean lattices as vector spaces

3.1 Proposition, lattices are vector spaces

$\mathbf{2} \equiv \{0,1\}$ is a field (the simplest there is) and $\mathbf{2}^4$ is a vector space over **2** with dimension $\left[\mathbf{2}^4 : \mathbf{2}\right] = 4$.

Since $\mathbf{2}^4$ is a vector space, there must exist an alternative description of a finite Boolean lattice that makes this transparent.

3.2 Boolean rings

Every Boolean lattice is isomorphic to a *Boolean ring*, that is, an idempotent[4] ring with unity. A Boolean ring is a structure $(B,+,\cdot,\mathbf{0},\mathbf{1})$[5] in which it makes sense to talk of addition and multiplication of elements $p,q \in B$. The Boolean sum, $p+q$, is called the *symmetric difference* of *p* and *q*.

3.3 Transformations of Boolean algebra and ring

Boolean lattice (algebra) $\rightarrow$ Boolean ring

$p \cdot q = p \wedge q$

$p + q = (p \wedge q') \vee (p' \wedge q)$

Boolean ring $\rightarrow$ Boolean lattice (algebra)

---

[3] Being "distinguished" does not mean that they are called into existence by the act of distinguishing them; they exist in any lattice, but we distinguish them by naming them.

[4] An operation $\circ$ on set *X* is said to be *idempotent* if $x \circ x = x$ for all $x \in X$.

[5] Formal axiomatisation of a Boolean ring: (B1) Associativity of addition: $x + (y+z) = (x+y) + z$; (B2) Associativity of multiplication: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; (B3) Commutativity of addition: $x + y = y + x$; (B4) Identity law for addition: $x + \mathbf{0} = x$; (B5) Identity law for multiplication: $x \cdot \mathbf{1} = x$; (B6) Distributive laws: $(a)\ x \cdot (y+z) = x \cdot y + x \cdot z$, $(b)\ (y+z) \cdot x = y \cdot x + z \cdot x$; (B7) Additive inverse law: $x + x = \mathbf{0}$; (B8) Idempotency: $x \cdot x = x$

$p \wedge q = pq$	meet
$p \vee q = p + q + pq$	join
$p' = 1 + p$	complement

3.4 Result

Every finite vector space has a basis

Since $\mathbf{2}^4$ is a vector space, there is a basis for it. There are many ways of obtaining a basis of independent vectors in this space, but the set $\{p, p', q, q'\}$ is <u>not</u> a basis. This is because $p + p' + q = 1 + q = q'$. This indicates that the set $\{p, p', q, q'\}$, corresponding to the logical $\{p, \neg p, q, \neg q\}$ is <u>not</u> its most informative substructure. Either of the following two related substructures may be chosen as a basis: -

The atoms

$$p \wedge q \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad p \wedge q' \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad p \wedge q' \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad p' \wedge q' \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The co-atoms

$$p \vee q \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad p \vee q' \equiv \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad p \vee q' \equiv \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad p' \vee q' \equiv \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

We see that $\mathbf{2}^2$ has a complete set of *atoms*.

3.5 Definition, atom (provisional)

In this context, a set of atoms for a finite Boolean lattice is a finite set of linearly independent vectors that forms a complete basis for it.[6]

It will be useful to have a distinct notation for these atoms and I shall use the symbols $\alpha_1, \alpha_2, \ldots$ to denote them. Here $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{p \wedge q,\ p \wedge q',\ p' \wedge q,\ p' \wedge q'\}$.

3.6 Result, finite atomic lattice

Every finite Boolean lattice has a set of atoms, and for this reason finite Boolean lattices are said to be *atomic*.

[6] An atom is formally defined below at 5.8 of this chapter.

Multiplication in the Boolean ring corresponds to a test for independence, and hence orthogonality[7]. For any two atoms we have $\alpha_i \cdot \alpha_j = \alpha_i \wedge \alpha_j = \mathbf{0}\ (i \neq j)$. In lattice theory in general the base structure from which the whole lattice may be constructed is called a *skeleton*.[8] In a Boolean algebra the set of atoms constitute its skeleton. Using the basis of atoms we see that every element of a Boolean lattice may be written uniquely as a join of atoms. We see that the set of atoms comprising the skeleton just is the partition of the space on which the lattice is defined.

4 Finite representation theory

A second series of correspondences has already been introduced by the partition $\mathbf{1} = \{1,2,3,4\}$ of the logical space. This is the correspondence between the algebraic operations $\wedge, \vee, '$ and the set-theoretic operations $\cap, \cup, '$. We have: -

$\wedge \leftrightarrow \cap \qquad \vee \leftrightarrow \cup \qquad ' \leftrightarrow '$

The atoms are mapped $\alpha_i \leftrightarrow \{i\}$, where *i* is a numerical label of the content of a partition of the underlying space. This means that every Boolean algebraic operation and corresponding logical operation also corresponds to an operation on the underlying set of all subsets of the partition of the space. For example,

$\{1,2,3\} \cap \{1,3,4\} = \{1,3\} \qquad \leftrightarrow \qquad (p \vee q) \wedge (\neg p \vee q) \equiv q$

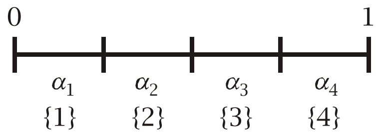
The partition of the underlying space turns it into a discrete topological space, and the atoms also comprise a topological basis for it. This can be visualised on our original diagram for that underlying space: -

$p \wedge \neg q$ {2}	$\neg p \wedge \neg q$ {4}
$p \wedge q$ {1}	$\neg p \wedge q$ {3}

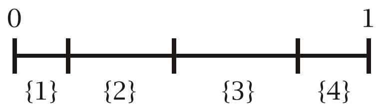
[7] Intuitively, this means the vectors are perpendicular to each other; formally, it means that their scalar product is 0.

[8] A skeleton is a representative set from which the lattice can be constructed. Davey and Priestley [1990] p.165 write: "Given any lattice *L* we … seek a representing set or 'skeleton' *P* from which to reconstruct *L*. We should like *P* to be a subset of *L* with the following properties: (i) *P* is 'small' and readily identifiable; (ii) *L* is uniquely determined by the ordered set *P*. Even more nebulously, we should also like: (iii) the process for obtaining *L* from *P* is simple to carry out." Atoms in a Boolean lattice meet these criteria. But, "if the lattice *L* is not required to be finite and Boolean, there may be too few atoms for the set of atoms to serve as a skeleton."

This diagram is a heuristic only, and as we know by means of space filling curves that any compact (closed, bounded) space may be mapped onto $\mathbf{1}=[0,1]\cong -\infty \cup \mathbb{R} \cup +\infty$ we see that the primitive structure that we are dealing with is a *partition of the continuum*, which is the *extended real line*, here represented most conveniently by $\mathbf{1}=[0,1]$. Thus, we may replace our heuristic diagram by: -

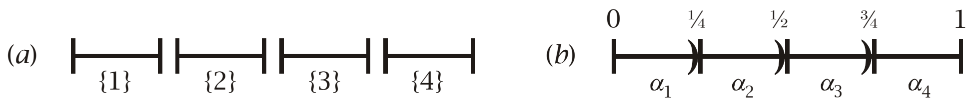


Let us provisionally treat each partition as an *atom* of space that cannot be further subdivided. In the above diagram I have shown each partition as being equal in length, but this measure of length is *extrinsic* to the structure, by which I mean to say that *viewed from the "outside"* the intervals in the partition may be any size. For example, this is what an external viewer might see: -



An observer from "inside" the space *intrinsically* would not be able to tell the difference between the two spaces, because he cannot *measure* the difference. In this model the interior content of each division is undifferentiated. A proposition affirms the whole of the division, without regard to individual content.

The lattice $\mathbf{2}^4$ is based on a partition of $\mathbf{1}=[0,1]$ that splits it into four disjoint, separated regions. If, indeed, we wish to explore the analytic logic of the continuum by this process, these regions may or may not be connected, but each sub-region may contain no point in common with the others. The following are partitions consistent with this underlying principle: -



In this second model we have intervals where, for example,

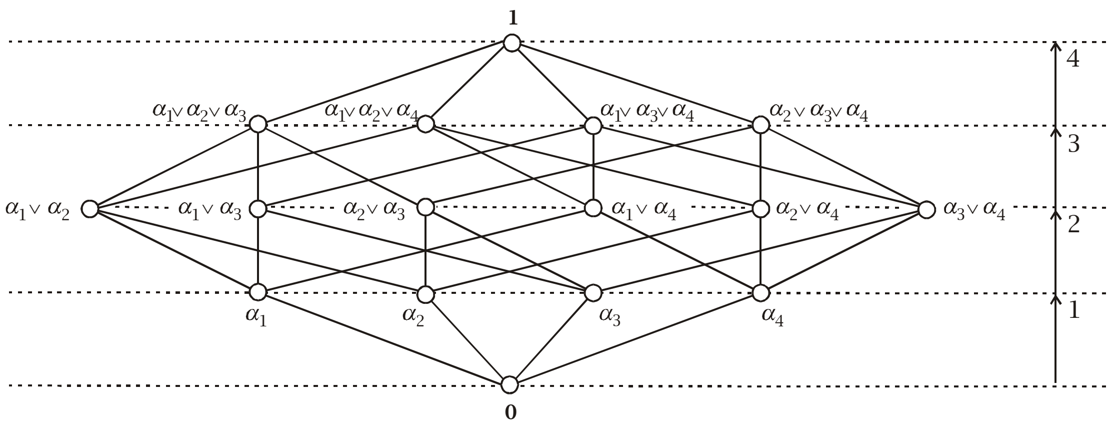
$$\alpha_1=\left[0,\tfrac{1}{4}\right),\ \alpha_2=\left[\tfrac{1}{4},\tfrac{1}{2}\right),\ \alpha_3=\left[\tfrac{1}{2},\tfrac{3}{4}\right),\ \alpha_4=\left[\tfrac{3}{4},1\right].$$

(This is just an illustration; other choices of the lengths and end-points of the interval are possible.) Observe in the second model that the first three partitions are half-open intervals – hence locally compact but not compact subspaces, whereas the final partition is a closed and bounded, i.e compact subspace. This observation shall be vital to all that follows as we extend the partition to an actually infinite partition of the continuum. [For compactness, see Chap. 2 Sec. 2.9.5 et seq.]

We have a choice to make as to how we interpret these intervals. (1) On the one hand, we could view each part as a whole whose contents make no contribution to the resultant algebra/logic, which is *discrete* and *atomic*. (2) Alternatively, if the content of each interval is included in the algebra then the partition ceases to be atomic, because the analysis of the space can be continued further. What then arises is an *interval algebra* that is *continuous* and *non-atomic*. [Chap. 5, 4.2 / 4.3] The endpoints of the partition constitute a *scaffold* or *skeleton* and the number of parts may be thought of as a *mesh* through which we sift the information that is contained in the interval $\mathbf{1} = [0,1]$. All partitions of the space into a discrete and finite mesh produce an atomic lattice, but there is always the possibility of a *finer* partition – that is, one that uses more partitions.

5 Boolean lattices as metric spaces

Not only is $\mathbf{2}^4$ a topological space but it is a metric space [Chap. 2, 2.9.10], equipped with an intrinsic unit of measure, which is based upon the "height" of the lattice point above **0**. The meaning of *height* is intuitively conveyed by the following version of the lattice $\mathbf{2}^4$, in which the points have been canonically written in terms of joins of atoms.



I also call height the *radial distance from* **0**. This is the intrinsic unit of measure in a Boolean algebra. For example the height of p in $\mathbf{2}^4$ is 2 (units) since $\mathbf{0} < p \wedge q < p$ is a shortest chain in $\mathbf{2}^4$ joining **0** to p of length 2. The distance of $p \vee q$ $(\alpha_1 \vee \alpha_2 \vee \alpha_3)$ from $\neg p \wedge q (\alpha_3)$ is also 2.

The formal definition of the metric requires a number of technicalities. If the reader is happy with the intuitive presentation above, he may omit the following: -

5.1 Definition, valuation

By a *valuation* on a lattice *L* is meant a real-valued function (functional) $v[x]$ on *L* which satisfies, (V1) $v[x]+v[y]=v[x\vee y]+v[x\wedge y]$. A valuation is *isotone* if and only if, (V2) $x\geq y$ implies $v[x]\geq v[y]$ and *positive* if and only if $x>y$ implies $v[x]>v[y]$.

5.2 Theorem

In any lattice *L* with an isotone valuation, the distance function $d(x,y)=v[x\vee y]-v[x\wedge y]$ satisfies for all $x,y,z,a\in L$

M1 $d(x,x)=0$ $d(x,y)\geq 0$ $d(x,y)=d(y,x)$

M2 $d(x,y)+d(y,z)\geq d(x,z)$ [triangle inequality]

M3 $d(a\vee x,a\vee y)+d(a\wedge x,a\wedge y)\leq d(x,y)$

5.3 Definition, cover

For *a, b* in *P* we say *b covers a*, or *a* is *covered* by *b*, if $a<b$ and whenever $a\leq c\leq b$ it follows that $a=c$ or $b=c$. We use the notation $a\ll b$ to denote *a* is covered by *b*. We also say *b* is the *immediate successor* of *a*, and *a* is the immediate predecessor of *b*.

5.4 Theorem

Any finite nonempty subset *X* of a poset has minimal and maximal members.

> <u>Remark</u>
>
> In chains, the notions of minimal and least (maximal and greatest) element of a subset are effectively equivalent. Hence: -

5.5 Theorem

Any finite chain has a least (first) and greatest (last) element.

5.6 Theorem

Every finite chain of *n* elements is isomorphic with the ordinal number ***n*** (the chain of integers 1, … *n*).

> <u>Remark</u>
>
> This ordinal *n* is *length* of the chain.

5.7 Definition, length

The length $l(P)$ of a poset *P* is the least upper bound of the lengths of the chains in *P*.

5.8 Definition, height, atom

In a poset P of finite length with $\mathbf{0}$, the *height* or dimension $h[x]$ of an element $x \in P$ is, by definition, the least upper bound (supremum) of the lengths of the chains $\mathbf{0} = x_0 < x_1 < ... < x_l = x$ between $\mathbf{0}$ and x. If P has a universal upper bound I then clearly $h[I] = l[P]$. Clearly also, $h[x] = 1$ if and only if x covers $\mathbf{0}$; such elements are called *atoms* or *points* of P.

5.9 Result

A nonzero element a of a lattice, L, is an *atom* iff for all elements $x \in L$ we have $x \le a \Rightarrow x = a$ or $x = 0$.

5.10 Definition, graded posets

A graded poset is a poset P with a function $g : P \to \mathbb{Z}$ from P to the chain of integers in their natural order such that

G1 $x > y$ implies $g[x] > g[y]$ strict isotonicity

G2 If x covers y, then $g[x] = g[y] + 1$

5.11 Definition, Jordan-Dedekind chain condition

All maximal chains between the same endpoints have the same finite length.

5.12 Lemma

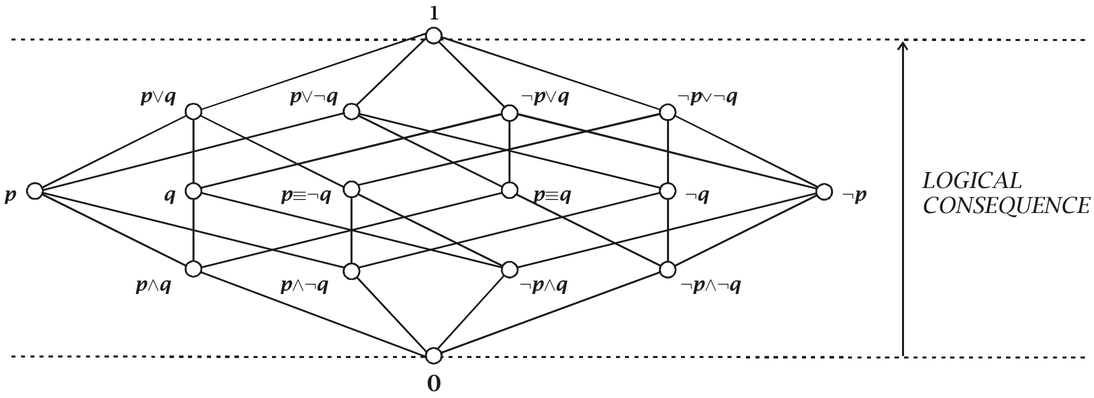
Let P be any poset with O in which all chains are finite. Then P satisfies the Jordan-Dedekind chain condition if and only if it is graded by $h[x]$.

5.23 Result

In any finite Boolean lattice height is a valuation that induces a distance function that acts as a measure on the Boolean lattice.

6 The relationship between logic and the lattice

There is a close relationship between a lattice and its logic; the perspective I am adopting here is that conceptually it is the lattice that comes first, so that the *logic is built over the lattice*. This is the same perspective as that of Aristotle who builds syllogistic logic over the relation of part to whole, which, thanks to the work of Birkhoff [1940] we now know to define a lattice. The primary relation of logic is *consequence*; the notion that one proposition *logically forces* another. Analytical logic is an application built over the fundamental properties of the lattice, that is, the relation of part to whole. In order to create analytical logic we must add an *extrinsic* sense of direction to the lattice – an *up* and a *down*. Propositions are mapped to lattice points and by convention any proposition that *lies below* another *implies* those propositions that *come above it* in the lattice.



The relation of consequence is denoted by $\vDash$; the expression $\gamma \vDash \psi$ may be interpreted to mean that the lattice point named by γ *lies below* the lattice point named by ψ and is connected to it. In this relation γ is called the *premise* and ψ the *conclusion*. This relation is clarified by the concepts of *filter* and *ideal*: every lattice point (proposition) that lies in the filter of a lattice point (proposition) γ is implied by ψ; every lattice point that lies in the ideal of a lattice point λ implies ψ. [See 6.1 below.] We identify filters by tracing lines in the lattice diagram upwards and ideals by tracing such lines downwards. We see that in the relation $\gamma \vDash \psi$ that in fact *the whole filter* is the conclusion of the premise ψ, so we may describe this logic as *single premise, multiple conclusion*. The logical proposition $p \wedge q$ corresponds in $\mathbf{2}^4$ to the lattice point α_1 and the partition $\{1\}$. The filter of $p \wedge q$ comprises every lattice point where $\{1\}$ is a subset. Thus, the filter contains: -

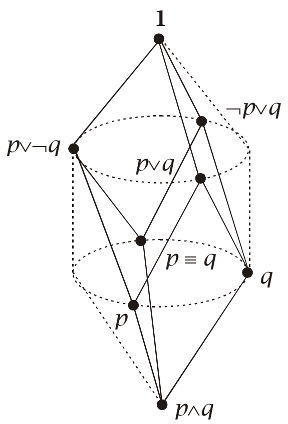
$\{1\} \leftrightarrow p \wedge q$

$\{1,2\} \leftrightarrow p$ $\quad$ $\{1,3\} \leftrightarrow q$ $\quad$ $\{1,4\} \leftrightarrow p \equiv q$

$\{1,2,3\} \leftrightarrow p \vee q$ $\quad$ $\{1,2,4\} \leftrightarrow p \vee \neg q$ $\quad$ $\{1,3,4\} \leftrightarrow \neg p \vee q$

$\mathbf{1}$

This filter may be visualised as follows: -



Based on these relations in the filter we may write, for example, $p \wedge q \vDash p$, $p \wedge q \vDash p \vee \neg q$ and so on. The ideal of $p \vee q$ comprises: -

$p \vee q \leftrightarrow \{1,2,3\}$		
$p \leftrightarrow \{1,2\}$	$q \leftrightarrow \{1,3\}$	$p \equiv \neg q \leftrightarrow \{2,3\}$
$p \wedge q \leftrightarrow \{1\}$	$\neg p \wedge q \leftrightarrow \{2\}$	$p \wedge \neg q \leftrightarrow \{3\}$
$\mathbf{0}$		

Formally, an ideal is defined as follows:-

6.1 Definition, ideal

An *ideal* is a nonvoid subset M of a lattice (or join-semilattice) L with the properties

1 $p \in M$, $x \in L$, $x \le p$ imply $p \in M$

2 $p \in M$, $q \in M$ imply $p \vee q \in M$

The *filter* is the dual [see 2.5 above] concept. Davey and Priestley (Davey and Priestley [1990] p. 13) introduce the helpful terms *up-set* and *down-set* for filter and ideal respectively. These are useful because they help one visualise the meaning. They also use the terms *increasing set* and *order filter* for filter, and *decreasing set* and *order ideal* for ideal.

The essential concept of logic is that of *proof path*, *formal derivation* or *deduction*. This is symbolised by $\vdash$. With this concept we encounter *formal analytic logic* proper. The relation of consequence is a property of a lattice endowed with an extrinsic sense of direction (an "up" and a "down") that gives meaning to the statement ψ is a consequence of γ. With $\vdash$ we encounter the idea of a *demonstration* of this by a finite deduction according to rules. Thus $\vdash$ is associated with a system of rules of inference each of which allows one to navigate upwards in the lattice along a path contained in a filter. Our example here is the lattice $\mathbf{2}^4$, which, being finite, has no need of quantifiers; we have seen that every one of its lattice points can be labelled uniquely in terms of the atoms $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ using only the join, $\vee$. Hence, we only need *one* rule of derivation, which is a rule for the introduction of the symbol $\vee$.

6.1 Rule for the introduction of disjunction

Let γ be any proposition corresponding to a point of the lattice; let ψ be any other proposition corresponding to a point of the lattice. Then

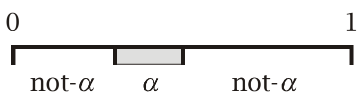
$$\frac{\vdash \gamma}{\vdash \gamma \vee \psi}$$

shall *mean* that from γ we may infer the proposition $\gamma \vee \psi$. This is also written, $\gamma \vdash \gamma \vee \psi$.

When $\psi \equiv \mathbf{0}$ then we derive trivially the law of identity, $\gamma \vdash \gamma$, which is also written, $\vdash \gamma \supset \gamma$, so in this way we confirm the classical description of analytical logic as *whatever follows from the law of identity*. This rule suffices not only for $\mathbf{2}^4$ but for every finite Boolean lattice, $\mathbf{2}^n$ where $n \in \mathbb{N}$. [See finite representation theorem, 10.1]

We see from this description that the negation sign is not formally required by classical propositional logic *founded on atoms*. Also the proposition $0 = 1$ would correspond to a collapse of the interval $[0,1]$ to a single point. Since our analytic logic is based upon a partition of the space $[0,1]$, $0 = 1$ is not a possible conclusion of any analytical logic. Since negation is also inessential in any finite system, *a fortiori*, $0 \neq 1$ is not a possible conclusion of any finite analytical logic. Analytical logic derives its consistency from the partition of the interval $[0,1]$ into disjoint segments, here called atoms, such that no two atoms can contingently coexist in the sense of a lattice meet (conjunction). Therefore, the consistency of any analytical logic derives from the consistency of the relation of part to whole, which is a synthetic relation extrinsic to any formal system. Consistency can only be a property of a formal system in a derivative sense, since in its primary sense it originates from the geometric intuition of the relation of part to whole, which is not a property of formulae, though formulae in a sound system of analytic inference must reflect it. In a system of deduction no rule may be allowed to introduce an inconsistent statement; the rules must be *sound*.[9] Since our system at present has only one rule, which is sound, and no symbol for negation, this system is sound overall. Therefore, by geometric intuition, it is also consistent.

It may be a surprising observation that classical propositional logic based on atoms requires only one logical connective and one rule of inference, so at this juncture I shall digress to discuss why other logical connectives and rules of inference are introduced. Firstly, let me make an observation about the atoms in a finite Boolean lattice – conceptually, these do not exclude negation, but on the contrary embody it. For an atom, α, marks off a segment of the space $[0,1]$, and partitions it into a part that affirms $\vdash \alpha$ and a part that affirms its complement α' which is denoted by $\vdash \neg\alpha$.



The transformation from a system of generators, $\{p, \neg p, q, \neg q\}$, to a system of atoms: -

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{p \wedge q,\ p \wedge q',\ p' \wedge q,\ p' \wedge q'\}$$

is akin to a change of basis of a vector space, and has the advantage of rendering transparent the underlying simplicity and constructible nature of the system. Both historically and

[9] A formal system of deduction shall be sound iff no deduction permits us to infer a contradiction. This is also expressed by $\vdash \phi \rightarrow \models \phi$ then the logic is **sound** .

psychologically speaking formal logic has two origins: (1) in the geometric intuition of the relation of part to whole that underpins the syllogism, and (2) in natural language inferences, whose connectives are *and, or, not* and *if… then.* It is the very essence of formal logic to *force* natural language inference into the frame of geometric intuition, and to treat those connectives as if their meaning were wholly expressed by the spatial relation of part to whole. The fruitfulness of this idea has been like a sea *sans bound*; as a generality, it is false.

In an application involving the mapping of natural language inferences onto a formal system, it is understandable that we should seek to build a formal system that embodies the rules of inference of natural language, albeit unconsciously influenced by the spatial analogy: such are the systems of Gentzen[10]. Furthermore, in this endeavour, we do not *begin* with the atoms and the relation to the atoms is not perspicuous; we begin, rather, with propositions that are equivalent to joins of atoms. Without a negation symbol we cannot get outside the filter generated by a proposition *p*; similarly, we need conjunction in order to move down the lattice in the direction of the atoms. Hence, the language is expanded from just comprising $\vee$ to include $\neg$ and $\wedge$. The construction, *if … then*, lies at the very core of human ability to attach ideas to one another and "flow" through them; hence, we add $\supset$ to the system, and also, equivalence $\equiv$.

With the symbol $\supset$ we encounter a real stress between the two systems of (1) natural and (2) formal language; these are reflected in the *paradoxes of material implication.*[11] In modus ponens: "If *p* then *q*, *p*, therefore *q*" we *seem* to *produce q* by following through *a train of thought* that starts with *p* and leads beyond it to *q*. This *may* be true of natural language but certainly is *not true* of formal language, where there is strictly no *train of thought* whatsoever. It is important to understand that in formal analytical logic, "if *p* then *q*", represented by $p \supset q$ means *nothing more* than $\neg p \vee q$ and therefore merely denotes a lattice point. Hence the inference $p \supset q,\ p \vdash p \wedge q$ becomes a statement of equivalence of names of a lattice point: -

$$\begin{aligned}(p \supset q) \wedge p &\equiv (\neg p \vee q) \wedge p \\ &\equiv (\neg p \wedge p) \vee (p \wedge q) \\ &\equiv p \wedge q\end{aligned}$$

---

[10] Gentzen produced systems of "natural deduction" and "sequent calculi" that were ostensibly modelled on natural language inferences. See Gentzen [1969] and Ungar [1945]. I shall not dwell on this point in this paper, save to express my view that his theory was unconsciously influenced by the nature of analytic logic and the results are *not* true reflections of natural language inferences.

[11] The "paradoxes" of material implication are (1) $\vdash p \supset (\neg p \supset q)$; (2) $\vdash p \supset (q \supset p)$. These are "paradoxes" because their meaning strikes one as odd. Likewise, there are inferences involving equivalences that are "paradoxical"; for example, $p \wedge q \vdash p \equiv q$. The mere fact that *p* and *q* are contingently given entails that they are equivalent. This strikes one as odd. The problem is resolved by realising that the equivalence asserted of *p* and *q* is contingent extensional equivalence; and that it is not asserted that *p* and *q* have the same meaning, or are equivalent in all possible worlds. Another "paradox" is, $\vdash (p \supset q) \vee (q \supset p)$. As a statement of "meaning" this is absurd. However, its extensional equivalent is, $\vdash (\neg p \vee q) \vee (\neg q \vee p)$, which is a statement of the law of excluded middle.

Then the statement $p \supset q,\ p \vdash q$ is equivalent to $p \wedge q \vdash q$, which in lattice terms is valid because q lies in the filter generated by $p \wedge q$. Likewise, if $p \wedge q$ represents a disjunction of atoms, $p \wedge q \equiv \alpha_{i_1} \vee \alpha_{i_2} \vee \ldots \vee \alpha_{i_n}$, then q is a *dilution of this statement* by the addition of further atoms to the disjunction, $q \equiv \left(\alpha_{i_1} \vee \alpha_{i_2} \vee \ldots \vee \alpha_{i_n}\right) \vee \left(\beta_{j_1} \vee \beta_{j_2} \vee \ldots \vee \beta_{j_m}\right)$.

> 6.2 (+) Principle of dilution
> *All deductive inference in analytic logic proceeds only by dilution of content; we can never reach a statement of generality greater than that already encompassed by the premises.*

This is why the logic is *analytic* and is precisely why it becomes absurd to suggest that arithmetic could be a manifestation of some form of analytic logic. About this problem, Poincaré wrote: -

> If … all the propositions [of mathematics] may be derived in order by the rules of formal logic, how is it that mathematics is not reduced to a gigantic tautology? The syllogism can teach us nothing essentially new, and if everything must spring from the principle of identity, then everything should be capable of being reduced to that principle. Are we then to admit that the enunciations of all the theorems with which so many volumes are filled, are only indirect ways of saying that A is A? (Poincaré [1982], p.394)

7 The vacuity of analytic logic

For any lattice there is no single name of a given lattice point: if p and q are names of lattice points, then $\neg p \vee q$ is name of *another* lattice point; but that point is also named by $p \supset q$ among others. The collection of all names of a lattice point is an *equivalence class*; where we write $\neg p \vee q$ we should write $[\neg p \vee q]$ to denote the equivalence class of which $\neg p \vee q$ is its representative and $p \supset q$ is another member. The Boolean algebra generated by these classes is called the *Tarski-Lindenbauum algebra* (See also Mendelson [1979], p. 43.); this is the Boolean lattice/algebra that we have been working with all along, it being tacitly assumed that the names we give to lattice points are just representatives of those points. The Tarski-Lindenbaum algebra is also known as the algebra of statement bundles.[12] [See Chap. 5, Sec. 5]

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[12] Any set of sentences is not closed under the operations of conjunction and disjunction and therefore does not form a Boolean algebra. (Mendelson Elliott [1970] p.160 et seq.) The conjunctions $p \wedge q$ and $q \wedge p$ are not identical. Therefore, they do not define unique joins in a lattice. So the set of sentences, *per se*, is not a Boolean algebra. This may be a surprising result, since it is natural to think of the set of sentences under the logical operations of conjunction, disjunction and so forth as a Boolean algebra. This difficulty is circumvented by the following definition of equivalence classes on the set of sentences: Define the equivalence class of the sentence p by $[p] = \{q : q \equiv p\}$. That is, as the set of sentences logically

Poincaré's claim – that to say all mathematics is analytic reduces mathematics to "a gigantic tautology" – is worth illustrating further. Firstly, the canonical name of any tautology is **1**, the "highest" point in the lattice, for which we have $\vdash \mathbf{1}$. Let $[\mathbf{1}]$ denote the equivalence class of the **1** element in a Boolean algebra. Members of $[\mathbf{1}]$ are to be called *names* of **1**. It is the multiplicity of these different names that conceals the essentially vacuous nature of analytic logic. Take, for instance, the tautology $\vdash (p \wedge (p \supset q)) \supset q$, which does not *look* vacuous: -

$\vdash (p \wedge (p \supset q)) \supset q$
$\vdash (p \wedge (\neg p \vee q)) \supset q$
$\vdash \neg(p \wedge (\neg p \vee q)) \vee q$
$\vdash \neg((p \wedge \neg p) \vee (p \wedge q)) \vee q$
$\vdash \neg(p \wedge \neg p) \vee \neg(p \wedge q) \vee q$
$\vdash \neg p \vee p \vee \neg p \vee \neg q \vee q$
$\vdash \mathbf{1}$

So $\vdash (p \wedge (p \supset q)) \supset q$ is just a diluted form of the law of excluded middle.

7.1 (+) Theorem, canonical representation of names of 1

Each name of **1** has a canonical representation of the form

$\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1}$ or $\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1} \vee \phi$

where $\# = \wedge$ or $\vee$ and $\phi = p_1 \vee p_2 \vee \ldots \vee p_n$, and each p_i is a contingent proposition. In the expression $\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1}$ each **1** is a separate name for the law of excluded middle. For example, $\mathbf{1} \vee \mathbf{1} = (p \vee \neg p) \vee (q \vee \neg q)$. We can allow duplication of the same irreducible proposition; for example, $\mathbf{1} \vee \mathbf{1} = (p \vee \neg p) \vee (p \vee \neg p)$.

<u>Proof</u>

The absorption laws for a Boolean algebra gives: -

$\phi \vee \mathbf{1} \equiv \mathbf{1}$ $\qquad \mathbf{1} \vee \mathbf{1} \equiv \mathbf{1}$ $\qquad \mathbf{1} \wedge \mathbf{1} \equiv \mathbf{1}$

Eliminate material implication in favour of joins and meets; i.e $p \supset q =_{df} \neg p \vee q$. Then the language of the propositional calculus contains only the symbols $\wedge$ (meet) and $\vee$ (join). Now consider a wff α in the language such that $\vdash \alpha$. Suppose α is not of the form $\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1} \vee \phi$. That is $\alpha \not\equiv (\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1} \vee \phi)$. By a tautology, $\alpha \equiv \neg(\mathbf{1}\#\mathbf{1}\#\ldots\#\mathbf{1} \vee \phi)$. Then: -

---

equivalent to *p*. These equivalence classes are also called *statement bundles*. With this definition we may obtain the following result: the set of statement bundles is a Boolean algebra.

$$\begin{aligned} & \vdash \alpha \\ \Rightarrow \quad & \vdash \neg(\mathbf{1}\#\mathbf{1}\# \ldots \#\mathbf{1} \vee \phi) \\ \Rightarrow \quad & \vdash \neg(\mathbf{1} \vee \phi) \\ \Rightarrow \quad & \vdash \neg\mathbf{1} \wedge \neg\phi \end{aligned}$$

Since we always have $\vdash \mathbf{1}$, this is a contradiction.

For a second version of this idea, let α be a wff in the language such that $\vdash \alpha$. Then, $\vdash \alpha \Rightarrow \vdash \alpha \wedge \mathbf{1} \Rightarrow \vdash \mathbf{1} \Rightarrow \alpha \equiv \mathbf{1}$. To extend this result to the predicate calculus, quantifiers must be eliminated in favour of infinite lists, $(\forall x)(\phi x) \rightarrow p_1 \wedge p_2 \wedge \ldots \wedge p_\lambda$ where λ is *any* ordinal, finite or transfinite. (It is a result that the class of all ordinals can be well-ordered.) Once we have eliminated quantifiers the only symbols remaining are $\wedge$ (meet) and $\vee$ (join) and the preceding argument applies.

7.2 (+) Theorem

$\mathrm{Eq}(\mathbf{1})$ is a proper class.

Proof

Given the elimination of quantifiers [Chap. 7, 4.4] in favour of infinite lists in conjunctive or disjunctive normal form, then the lists can have length of any ordinal. Since the class of all ordinals is proper [Defined, Chap. 2 / 1.3.5], so then is $\mathrm{Eq}(\mathbf{1})$.

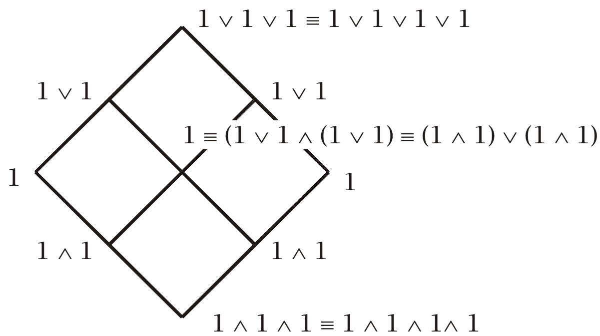
A t*autology* is a statement in canonical form, $\mathbf{1}\#\mathbf{1}\# \ldots \#\mathbf{1}$ or $\mathbf{1}\#\mathbf{1}\# \ldots \#\mathbf{1} \vee \phi$, where the list is finite. A *logical truth* is either a tautology or a statement of canonical form, $\mathbf{1}\#\mathbf{1}\# \ldots \#\mathbf{1}$ or $\mathbf{1}\#\mathbf{1}\# \ldots \#\mathbf{1} \vee \phi$ where the list is infinite.

7.3 (+) Example

Rule of instantiation, $\vdash (\forall x)(\phi x) \rightarrow \vdash \phi a$, when quantifiers are eliminated becomes,

$\vdash p_1 \wedge p_2 \wedge \ldots \wedge p_\lambda \rightarrow \vdash p_i$.

The differing names of **1** form an equivalence class and members of it also form a lattice, part of which is shown thus: -



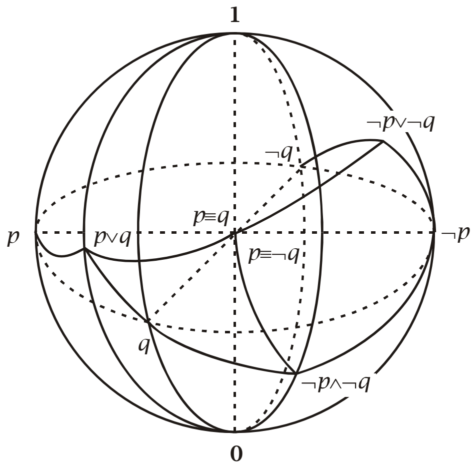
7.4 (+) Definition, pseudo-lattice

The lattice above shall be called the *pseudo-lattice of* **1**.

It is a pseudo-lattice because all the individual lattice points here are all names of the same lattice point in the lattice defined by the partition of some space into atoms.

8 The global geometry of analytic logics

There is another representation of $\mathbf{2}^4$ showing it as related to a manifold embedded in 4-dimensional Euclidean space, E^4.



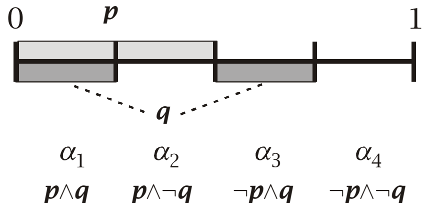
In this diagram we should give particular attention to the lattice points: -

$p \equiv q \quad \Leftrightarrow \quad (p \wedge q) \vee (\neg p \wedge \neg q) \qquad\qquad p \equiv \neg q \quad \Leftrightarrow \quad (p \wedge \neg q) \vee (p \wedge \neg q)$

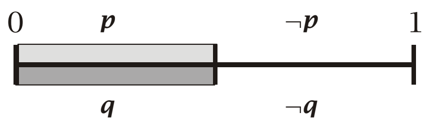
These are shown at the centre of the diagram, but they represent distinct points, so that there are two "interpenetrating sheets" dispanded throughout the interior of this "topological sphere" and three-dimensions is insufficient to embed this structure. This structure, nonetheless, is a manifold, notwithstanding the fact that it is defined by the points on it rather than the sheets, which would appear to make it discrete.[13] The reason for this is that it is generated from a partition of the closed real line $[0,1]$, which is a manifold.[14]

[13] A differentiable manifold is continuous. Continuity is relative to what constitutes an open set in a topological space. Therefore, a structure that is discrete when viewed extrinsically may be continuous from an intrinsic perspective.

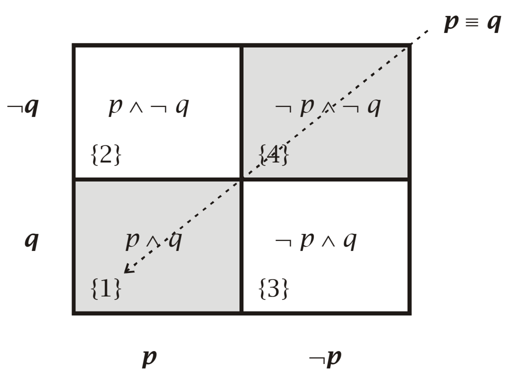
[14] The lattice $\mathbf{2}^4$ is a discrete structure, but it may be viewed as the framework of a structure whose interstices may be likened to a rubber sheet. Therefore, it is possible to define a differential geometry on this structure, and I conjecture that there exists a *differential geometry of discrete structures*. In my original investigations of the problem concerning the distinction between those structures that are effectively computable and those that are not, I aimed to discover that there are geometric invariants akin to those encountered in, for example, the *theorem egregrium* that would show that the global geometry of a structure to which a digital computer is bound is distinct from that to which human intelligence, as manifested through arithmetic, is bound. I still think this is a potentially fruitful area of research.



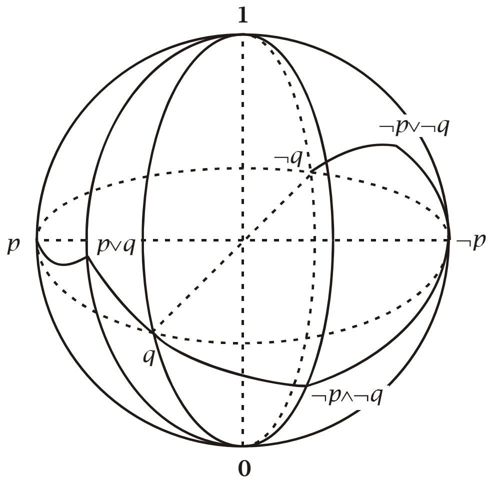
Since the underlying structure is a manifold, the structure generated from it is also a manifold. In the above diagram I have also shown those parts of the partition of $[0,1]$ that correspond to propositions *p* and *q*. In any logical application *p* and *q* represent contingent statements. For example *p* could represent "The crew is male" and *q* "The crew is going to Mars". Then $p \equiv q$ means that anyone going to Mars is male. This is a contingent statement because females could have been chosen for the expedition but weren't. Nonetheless, the effect is to render the distinction between *p* and *q* for *the given domain* (crew) unnecessary (here "for the expedition to Mars" and "male" are equivalent). In other words, if we view each proposition *p, q* as a dimension (they are independent vectors), then a state of affairs in which we have $p \equiv q$ *collapses the dimension.*



Another way of identifying this state of affairs is from our original description of the underlying space of $\mathbf{2}^4$.



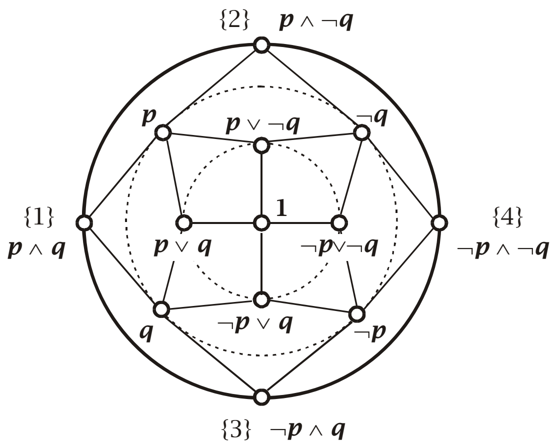
Whereas *p* and *q* are vertical and horizontal partitions of this space, $p \equiv q$ is a diagonal partition. If we eliminate all diagonal (and anti-diagonal) sets from our model of a lattice, we obtain a *shell.* For $\mathbf{2}^4$ this is:-



Assuming the points are pinned to a manifold, this has Euler characteristic: $\chi = V - E + F = 14 - 36 + 24 = 2$, making it a topological sphere. [Chap. 2, Sec. 2.11] The point **0** represents an absolute contradiction, so we can never affirm that. The *lowest level* of the model is that of the atoms, which in $\mathbf{2}^4$ are: -

$\alpha_1 \equiv p \wedge q$ $\alpha_2 \equiv p \wedge \neg q$ $\alpha_3 \equiv \neg p \wedge q$ $\alpha_4 \equiv \neg p \wedge \neg q$

Therefore, deleting **0** also from the manifold, we obtain as a model of the analytic logic of $\mathbf{2}^4$ as the *punctured sphere* or *disk*: -

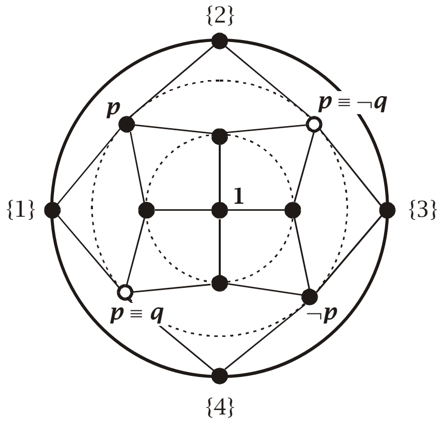


This has Euler characteristic: $\chi = V - E + F = 13 - 24 + 12 = +1$. As we expect, it is a topological disk.

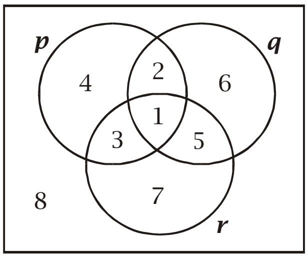
If now we return to the model in which we had the sphere with the addition of the vertices: -

$p \equiv q \quad \Leftrightarrow \quad (p \wedge q) \vee (\neg p \wedge \neg q) \qquad\qquad p \equiv \neg q \quad \Leftrightarrow \quad (p \wedge \neg q) \vee (p \wedge \neg q)$

we note that these add 2 vertices, 4 edges and 4 faces to the structure, thus giving Euler characteristic: $\chi = V - E + F = 16 - 40 + 28 = 4$. In the model of the punctured sphere (deleting **0**) we observe that we can replace q by $p \equiv q$ and $\neg q$ by $p \equiv \neg q$, giving another disk:



(By swapping one pair of the outer labels we obtain a new disk.) The model of $\mathbf{2}^4$ comprises two topological spheres, one lain "over" the other, and sharing the points shown filled above. This accounts for the Euler characteristic of 4 for the double-sphere model. These are two spheres[15], $S^1 \oplus S^1$ lain side by side and joined at all but 2 lattice points. Once the **0** is deleted, we have two disks $D^1 \oplus D^1$ as the model of the analytic logic $\mathbf{2}^4$. Examining a refinement of the partition of $[0,1]$; for example, when we have eight partitions (atoms).



In this diagram, for convenience, I have dropped the curly brackets, indicating the partitions by numbers 1,2,3, ... The atoms are: -

$\{1\}$	α_1	$p \wedge q \wedge r$	$\{2\}$	α_2	$p \wedge q \wedge \neg r$
$\{3\}$	α_3	$p \wedge \neg q \wedge r$	$\{4\}$	α_4	$p \wedge \neg q \wedge \neg r$
$\{5\}$	α_5	$\neg p \wedge q \wedge r$	$\{6\}$	α_6	$\neg p \wedge q \wedge \neg r$
$\{7\}$	α_7	$\neg p \wedge \neg q \wedge r$	$\{8\}$	α_8	$\neg p \wedge \neg q \wedge \neg r$

The truth functions are combinations of these. For example: -

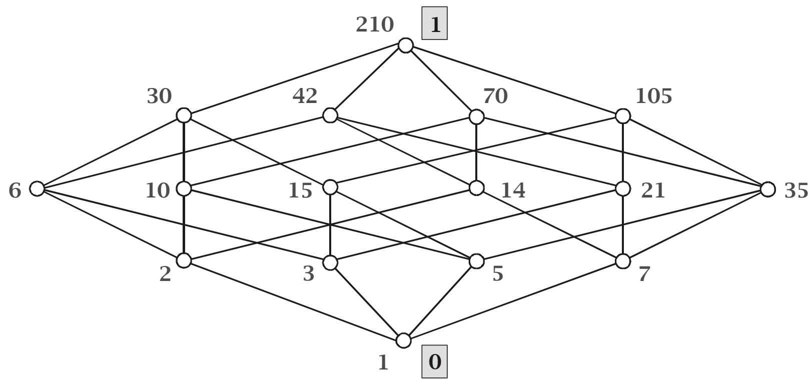
$\{1\}$	$p \wedge q \wedge r$	$\{1,2,3,4,5,6\}$	$p \vee q$
$\{1,2\}$	$p \wedge q$	$\{1,2,3,4,5,6,7\}$	$(p \vee q) \vee (\neg p \wedge \neg q \wedge r)$
$\{1,2,3\}$	$(p \wedge q) \vee (p \wedge r)$	$\{1,2,3,4,5,6,7,8\}$	1
$\{1,2,3,4\}$	p	$\{1,2,5,6\}$	q
$\{1,2,3,4,5\}$	$p \wedge (\neg p \wedge q \wedge r)$	$\{1,3,5,7\}$	r

[15] I am using $S^1 \oplus S^1$ to intuitively denote the special relationship between the spheres in this case. It would require further analysis to define the relationship formally.

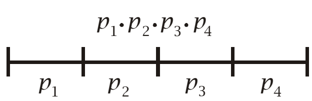
Each higher lattice point is a dilution of the information contained in the lattice point below it. The topology as a whole is still subdivided into disjoint structures. There are 256 lattice points, being all binomial coefficients of 8 symbols. In the disk diagram there will be seven concentric circles centred on 1 (the eighth); in each of the 7 there will be 8 lattice points. Each permutation of the symbols 1 to 8 defines a disk, but a number of these will be identical; it is a problem in combinatorics to determine the precise number of sheets.

9 Analytic logic does not encompass order

There is yet a further characterisation of the lattice $\mathbf{2}^4$ that derives from the observation that the set $\{1,2,3,5,6,7,10,14,15,21,30,35,42,70,105,210\}$ where $\vee$ = least common multiple and $\wedge$ = greatest common divisor defines a Boolean lattice[16].



This is just one among an infinite collection of number sets that define a lattice that is isomorphic to $\mathbf{2}^4$. This lattice gives rise to the *logic of the partition of the number 210*, which is based on the observation that a number may be likened to a space. For example, any number that is the unique product of 4 prime numbers: $\tau = p_1 \cdot p_2 \cdot p_3 \cdot p_4$ generates a lattice isomorphic to $\mathbf{2}^4$.



If the lattice is inverted (as we are allowed to do [Principle of duality 2.5 above], then we obtain a new lattice that embodies the logic of divisibility. For example, we have from x is divisible by 6 the conclusion x is divisible by 2, because 2 belongs to the filter defined by 6. In this inverted lattice, the prime numbers become prime ideals [Chap. 5, Def. 7.19].

[16] Let m be a positive integer. Let the structure A be given by

$$0_A = 1 \qquad 1_A = m \qquad p \wedge q = \gcd(p,q) \qquad p \vee q = \operatorname{lcm}(p,1) \qquad p' = \frac{m}{p}$$

Then A is a Boolean algebra iff m is not divisible by the square of any prime.

9.1 Example, logic of divisibility

This is an example of an inference in the logic of divisibility.

To prove: $(\forall n)(4|n \supset 2|n)$

1. Let $4|n$
2. $n = 4m$ for $m \in \mathbb{N}$
3. $n = 2 \times 2m$
4. $2|n$
5. $4|n \supset 2|n$
6. $(\forall n)(4|n \supset 2|n)$

This is equivalent to the claim that $2|n$ lies in the filter of $4|n$, or that

$2|n = \{0,2,4,...\}$ is a supraset of $4|n = \{0,4,8,12,...\}$.

> Reminder: All inferences in the lattice proceed by dilution. Thus, $p \supset q$ is only possible if q is a join of which p is a member. Thus $p \supset p \vee ...$ is the *fundamental rule of inference*. [See 6.2 above: the principle of dilution.]

Analytic logic is founded upon the concept of a partition of some space, and the relation of part to whole thereby generated. I observed above that the partition need not be uniform, uniformity of space being a concept extraneous to analytic logic, having some other synthetic origin. This self-evident conclusion follows immediately upon the observation that *nothing is required to be added to the concept of the partition in order to generate a complete analytic logic* – at least, not so far as the finite case is concerned, for we have as yet to examine the infinite one. Now I add the following additional observation: that *order is also extrinsic to this bare notion of an analytic logic founded upon a partition of space*. Any finite set may be well-ordered; and therefore, we may order the atoms of a partition and arbitrarily designate them as being in increasing order. The expressions of any formal language may also be well-ordered, and so we may order the labels of the atoms if we so wish. If indeed our intention is to *model the analytic logic of the continuum*, then we may regard our atoms as *atoms of space* and require them to be ordered in the sense of progressing increasingly from 0 to 1. In other words, we *impose* upon the partition *the additional structure* inherited from the synthetic and geometric intuition of *direction*. Analytic logic is concerned with cardinal numbers [Defined Chap.2, Sec. 2.7 et seq.] and shows that each cardinal forms a division ring; the uniqueness of prime factorisation turns each of these rings into a lattice. Nonetheless, the concept of an ordinal number makes no appearance at the ground level in the notion of an analytic logic, and, unless there is some addition arising from the consideration of lattices founded upon some infinite partition of $[0,1]$ any treatment of ordinal numbers must make its presence felt in some other way. All ordinal successions are founded upon the prime succession of the natural numbers and upon the synthetic act of counting that generate them. *Prima facie*, there is no reason to suppose that arithmetic is a form of analytic logic, to which I add the following caveat: it remains to be seen whether the embedding of set theory within first-order logic and the definitions of number thereby created serve to refute this conclusion, which I hereby propose as a temporary one based on the evidence so far presented.

9.2 About ordered pairs

As the concept of an ordered pair may be defined in terms of sets it may be objected that, notwithstanding the observations above, order is an analytic relation. This is a point of view commonly expressed in the literature: -

> Occasionally, one must resort to an artificial definition in order to "embed" some mathematical notion smoothly in ZFC. One such definition is Kazimierz Kuratowski's definition of the **ordered pair** of any two objects: $(x,y)=\{\{x\},\{x,y\}\}$. The set on the right side of this equation has no conceptual connection with ordered pairs. It is used simply because it allows us to prove, in ZF, the two essential properties of ordered pairs; that the ordered pair of any two sets exists, and that $(x,y)=(u,v)$ if and only if $x=u$ and $y=v$." (Wolf [2005] p. 75.)[17]

Applying Kuratowski's definition, we see

$(0,0)=\{\{0\},\{0,0\}\}=\{\{0\},\{0\}\}=\{\{0\}\}$ $\quad$ $(1,1)=\{\{1\},\{1,1\}\}=\{\{1\},\{1\}\}=\{\{1\}\}$

These equivalences indicate the arbitrary nature of the definition, since the connection between $\{\{0\}\}$ and $(0,0)$ is wholly conventional. Consider also: -

$(0,2)=\{\{0\},\{0,2\}\}=\{1,\{0,2\}\}$

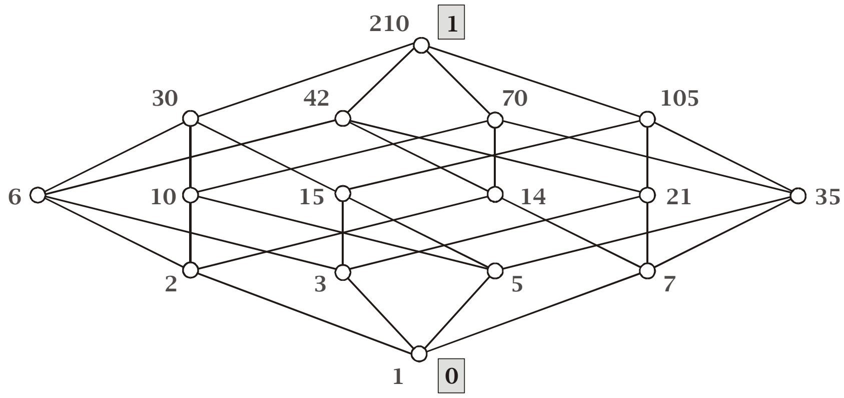
$$
\begin{aligned}
(0,0,0) &= (0,(0,0)) \\
&= \{\{0\},\{0,(0,0)\}\} \\
&= \{1,\{\{0\},\{0,(0,0)\}\}\} = \{1,\{1,\{\{0\},\{0,0,0\}\}\}\} = \{1,\{1,\{1,1\}\}\} = \{1,\{1,\{1\}\}\}
\end{aligned}
$$

I contend that the general "identical up to isomorphism" rule is *just false.*

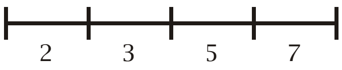
9,3 Order invariance

There is a third essential property of any ordered pair (x,y); namely, that *x* comes *first* and *y* *second.* But this statement "begs the question" since formalists must insist that only the formal properties of sequences define them. If we seek to embed the theory of ordinals in the theory of partitions (unordered sets), there must be some *invariant* method for constructing the ordinals from these partitions. Observe that one version of our lattice $\mathbf{2}^4$ is: -

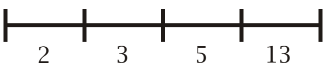
[17] There is a similar remark by Potter, who draws parallels with Benacerraf's problem [Chap. 16, Sec. 3]; he writes, "the ordered pair as it is used here is to be thought of only as a technical tool to be used within the theory of sets and not as genuinely explanatory of whatever prior concept of ordered pair we may have had." (Potter [2004] p. 65.)



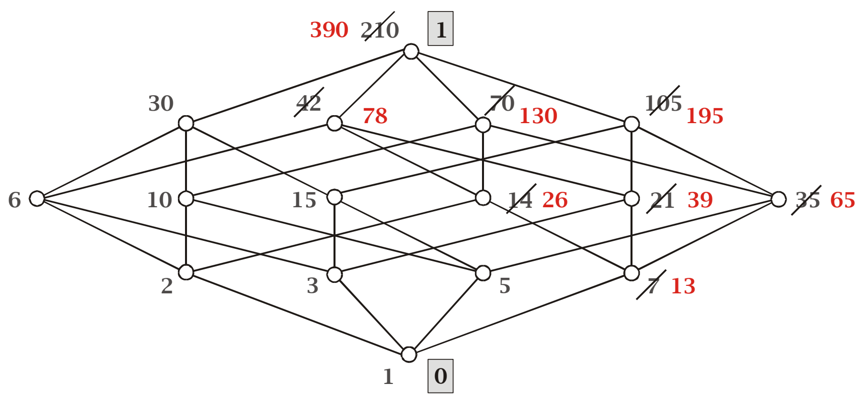
This is based on some disjoint partition of space



However this is achieved, the structure of the lattice does not depend in any way on the internal structure of the partitions 2, 3, 5, 7; it is wholly determined by the joins built upon it. Observe the middle line of the lattice and how the ordinals there run in the sequence 6, 10 ,<u>15, 14</u>, 21, 35. This is not an increasing sequence, but the lattice points may be rearranged so that it becomes an increasing sequence. If the structure of the ordinals built over the partition is invariant then we expect the same pattern to emerge for every other isomorphic partition of $\mathbf{2}^4$. However, the partition



Creates a lattice where the lattice points exhibit a different order: -



The middle row now runs 6, 10, <u>15, 26</u>, 39, 65 and the order of the two middle terms has been reversed. (If it is objected that the primes, 2, 3, 5, 13, in this case are not consecutive primes, then consider 17, 19, 23, 29, which are consecutive and where the order is also reversed:

$19 \times 23 = 437$, $17 \times 29 = 493$.) Therefore, isomorphic lattices do not give rise to an order invariant relation on the ordinals. Alternatively, if order invariance is insisted upon, then some lattices must be excluded and the logic as a whole ceases to be complete. A poset, *P*, can only be a lattice if for all $x, y \in P$, $x \vee y$ and $x \wedge y$ exist in *P*.

9.4 Definition, complete lattice

Let *P* be a non-empty ordered set. If $\bigvee S$ and $\bigwedge S$ for all $S \subseteq P$, then *P* is said to be a *complete lattice*.

The distinction between a lattice and a complete lattice only arises in the case of an infinite set; all finite lattices are both atomic and complete. When the lattice is infinite then to say it is complete is to allow that corresponding to every infinite collection *S* of distinct lattice points there is both an infinite meet and infinite join; that is what $\bigvee S$ and $\bigwedge S$ signify. One way in which to overcome the manner in which the above lattice inverts the order of the mid-points is to delete one of these points – say the one labelled 26. Then the structure ceases to be a lattice because $2 \vee 13 = 78$ and $2 \vee 13 = 130$; since it is not a lattice it is also not a complete lattice. Hence, if order invariance of the ordinals is insisted upon, the lattice must cease to be complete. This *a fortiori* makes the logic incomplete. [See below, section 10.4 and definition 10.5. Incomplete logics are the subject of Chapter 9.]

10 Finite Boolean representation theorem and the hierarchy of analytic logics

The properties of $\mathbf{2}^4$ described here are inherited by all finite Boolean algebras. This is the content of the finite Boolean representation theorem.

10.1 Finite Boolean representation theorem

Every Boolean algebra is isomorphic to a field of sets.

Informal proof

Using the notation of algebraic field extensions, we have $[\mathbf{2}:\mathbf{2}] = 1$, where $\mathbf{2} = \{0,1\}$ is any Boolean algebra of two elements. Every Boolean algebra $\mathbf{2}^{k+1}$ is a vector space over $\mathbf{2}$, and $[\mathbf{2}^{k+1}:\mathbf{2}] = 2 \times [\mathbf{2}^k:\mathbf{2}]$, since $\mathbf{2}^{k+1} = \mathbf{2} \times \mathbf{2} \times \ldots \times \mathbf{2}\ (k+1 \text{ times})$. Therefore, for all $n \in \mathbb{N}$, $[\mathbf{2}^n:\mathbf{1}] = 2^n$.

Each $\mathbf{2}^n$ has a unique basis of vectors that may be placed into one-one correspondence with the singleton sets of elements of some set *A* where $|A| = n$. We also have $|\mathbf{2}^n| = 2^n$.

This theorem is proven formally in chapter 5, section 5. We have started with the concrete example of the lattice $\mathbf{2}^4$. However, we see by the above inductive proof that the subject of our enquiry so far has been the class of all finite (Boolean) lattices: $\{B : B \cong \mathbf{2}^n, n \in \mathbb{N}\}$. This set is potentially infinite, not bounded above and not closed.

10.2 The hierarchy of analytic logics and reflections on Poincaré's thesis

The finite Boolean representation theorem uses the principle of complete induction in order to establish a theorem about *all finite Boolean lattices.* Though we have as yet to examine the case where there is an analytic logic based on an infinite partition of space, we must conclude that no analytical reasoning based on a finite partition of space could possibly produce the generality encompassed by the finite Boolean representation theorem, or any similar theorem that is based on complete induction. Regarding finite partitions, we see a hierarchy of analytic logics: -

$\mathbf{2}$	$\{\mathbf{0},\mathbf{1}\}$	Logic of $0 \equiv$ contracdiction; $1 \equiv$ necessary truth.
$\mathbf{2}^{2^1}$	$\{\mathbf{0},p,\neg p,\mathbf{1}\}$	Logic of one contingent proposition *p* and its negation.
$\mathbf{2}^{2^2}$	$\{\mathbf{0},p,\neg p,q,\neg q,\mathbf{1}\}$	Logic of two contingent propositions.
...	...	...
$\mathbf{2}^{2^n}$	$\{\mathbf{0}, \dots, \mathbf{1}\}$	Logic of *n* contingent propositions.
...	...	...

We encounter here the following principle – while analysis of the logic of *n* contingent propositions will produce the logic of *k* propositions for all $k \le n$, no amount of analysis will enable one to progress in the opposite direction to arrive at the logic of $n + k$ independent propositions for all $k > 0$. In other words, analytic logic is analytic on the "way down" and synthetic on the "way up".[18] The sequence of analytic logics, $\mathbf{2}^{2^0}, \mathbf{2}^{2^1}, \mathbf{2}^{2^2}, \dots, \mathbf{2}^{2^n}, \dots$ is *not*

---

[18] Leibniz attempted to demonstrate that arithmetic is analytic. His argument is refuted by Frege in his *Foundations of Arithmetic*. This is Leibniz's putative proof that $2+2=4$ is analytic. Definitions:

[1] $2 = 1 + 1$ [2] $3 = 2 + 1$ [3] $4 = 3 + 1$

Axiom: If equals are substituted for equals, the equality remains. Proof: -

$2+2 = 2+1+1 \text{ (by [1])} = 3+1 \text{ (by [2])} = 4 \text{ (by[4])}$

Therefore, by the axiom $2+2=4$. Frege's counter-argument: The definitions each rest on *primitive* notions. To clarify this, suppose in $2 = 1 + 1$, each 1 represents the same object. Then $'1+1'$ means '1 *and* 1', which is just 1. (Compare, adding the same point to the same point – you do not have two points, just one.) Alternatively, suppose each 1 represents a different object, and let us mark this in some way; thus, $2 = 1' + 1''$; now we cannot add the two separate 1's, or 2 is not uniquely determined if we do so. Suppose we have different 1's: $2 = 1^{\circ} + 1^{*}$ - we have determined a different object 2. Frege summarizes this objection succinctly: "If we try to produce the number by putting together different distinct objects, the result is an agglomeration in which the objects contained remain still in possession of precisely those properties which serve to distinguish them from one another; and so that is not the number. But if we try to do it in the other way, but putting together identicals, the result runs perpetually together into one and we never reach a plurality." (Frege [1980] p. 50.) This problem may be resolved as follows: In $'1+1'$ the *bounded above*. To achieve a generality relating to *all analytic logics whatsoever* there must be an additional principle of synthetic reasoning that enables one to generalise from the finite to the infinite case. This principle is supplied by complete induction which is required to prove every generality about finite Boolean lattices, including the finite Boolean representation theorem, above.

The theory of chains embeds complete induction as an instance of transfinite induction on transfinite ordinals; that is, we embed the sequence $n \in \mathbb{N}$ within a sequence that carries on into the transfinite, $\mathbf{2}^{2^{\alpha}}$ where $\alpha \in \mathrm{On}$ and On denotes the proper class [Chap. 2, Sec. 1.3.5] of all ordinals. This is a *proper class* that is unbounded above and though it is equipped with a *principle of transfinite induction*, we have an exact replication of the problem for the transfinite case as we have for the finite case. There is no structure whatsoever that could be defined in such a manner that we could, by means of analysis alone of that structure, deduce a principle of induction that is defined on the whole of that structure.

Analysis is based on the mental act of division a line or space into parts. This mental act is known to us primarily as a finite act of the mind producing a finite division. Suppose I divide a line into two parts, and then divide one of those parts again in two, and so on. *It takes an act of the mind to infer from the mere possibility that this division can be repeated indefinitely that there must be an infinite partition of the line.* This mental act is akin to the synthetic foundation upon which reasoning by induction is based. Therefore, any principle derived from transfinite analytic logic must also have a synthetic basis.

It is universally allowed by set theorists (a) that the proper class of all ordinals is not a set; (b) that the proper class of all sets is not a set; (c) that there is no set that can act as a model for all set theory; (d) though there may be sets that can act as models of parts of set theory, those parts would have to exclude axioms that permit the definition of transfinite induction – specifically the axiom of infinity. The axiom of infinity asserts *that there exists an infinite set*; how could that possibly be a principle of analytic reasoning? How could it be possible that by *analysis* of a finite structure alone one could arrive at the notion of the actual existence of a completed infinity? Therefore, the prospects for demonstrating that all mathematics is a species of formal analytic reasoning are dim. While it remains to work out thoroughly the mathematics of this situation to demonstrate conclusively that this is so, *prima facie* formalists must supply an answer to Poincaré's thesis.

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part $'+1'$ represents a *primitive notion* of taking the successor of a given number. In counting we reach the next successive number $n+1$ from a given number n by 'adding 1'. Thus, 'adding 1' is a primitive notion. However, while this clarifies the meaning of $2=1+1$, it does not overturn Kant's interpretation of that sum as a *synthetic* proposition; but rather emphasizes that the number 2 is the synthesis of the number 1 together with the operation of taking its successor. The numbers are given, not defined, and the operation of taking a sum is synthetic, not analytic. *No amount of analysis of the number 2 will ever reveal the number 3. The number 3 is not a part of the number 2.* Furthermore, as Frege points out, Leibniz's argument has a hidden premise. The line $2+2=2+1+1$ should be written $2+2=2+(1+1)$. From this line, it is a further proposition to step to $2+(1+1)=(2+1)+1$. This line assumes a general property of addition of numbers, namely, that they are *associative*: For all $a,b,c \in \mathbb{N}$ $a+(b+c)=(a+b)+c$. This is a further axiom that expresses a *primitive*, that is, *synthetic* property of numbers. Thus, Leibniz's argument fails to demonstrate that arithmetic is analytic.

10.3 About axioms

The theme of my investigation so far has been the relationship between any finite Boolean lattice $\mathbf{2}^n$ and the classical propositional logic built over it. The perspective here is that conceptually *it is the lattice that comes first* and subsequently the *logic is built over it*, in the sense that it is an application of the relations found in the lattice to the problem of formal inference. It is usual in other texts to reverse the order of presentation, to exhibit classical logic as a system in its own right; the connection to the lattice being sometimes obscure, though, to be sure, strongly forced upon one through the presentation of truth tables. Regular features of such a presentation are (a) a set of axioms or (b) a collection of rules of inference; (c) a proof of the deduction theorem; (d) a proof of the soundness and completeness of the system.

What are axioms and rules of inference and what is their relationship to the lattice and the points that the lattice contains? The main point to acknowledge here is that *axioms are not atoms*, and they do not appear as distinct lattice points. Taken collectively, as a single conjunction, the axioms constitute a name of the maximal point of the lattice, representing *tautology* or *logical truth*, and denoted by **1**. The rule of inference (modus ponens or the rule for $\vee$-introduction) embodies the principle that *whatever lies in a filter above a point in a lattice is held to be a necessary consequence of whatever statement that lattice point represents.* Let Σ represent a complete set of axioms for the propositional calculus; then $\Sigma \equiv \mathbf{1}$. The axioms of propositional logic make no reference to the size of the domain, which here means, the cardinality of the partition of $[0,1]$. It is a consequence of the definition of a Boolean lattice as a complemented distributive lattice that only partitions of cardinality 2^k for some $k \in \mathbb{N}$ give rise to a classical logic, but other degenerate forms that are distributive but not fully complemented are encompassed by this approach[19].

Every lattice point p has alternate name $p \wedge \Sigma \equiv p \wedge \mathbf{1} \equiv p$. This justifies the rule in a formal derivation that at any point an axiom may be introduced without assumption. The introduction of an atom affirming a proposition is equivalent to the rule of assumptions. In any application of the logic it is assumed that the affirmation of an atom is an affirmation of a contingent state of affairs, and that the logic then infers what must necessarily follow from what is initially given contingently. A *one step inference* is a single application of either the rule of assumption (to introduce a statement corresponding to a lattice point, and constituting a premise), the rule permitting the introduction of any axiom (without premise), or any rule of inference. A rule of inference corresponds to a step from one lattice point to another that immediately covers it. A *chain of deductions* is a chain of sound one step inferences.

Given two atoms $\alpha_i, \alpha_j, i \neq j$ the conjunct (meet) of these is a contradiction: $\alpha_i \wedge \alpha_j \equiv \mathbf{0}$; therefore, it is a rule that no two atoms can be conjoined in this way. (This is affirmed in the law of non-contradiction: $\neg(p \wedge \neg p)$, since $\alpha_i, \alpha_j, i \neq j$ entails $\alpha_j \supset \neg\alpha_i$.)

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[19] See below, section 10.7

Consequently, any sound inference of the form: $p, q \vdash \phi$ may be combined into a single statement: $p \wedge q \vdash \phi$, since *p* and *q* cannot be atoms. Therefore, classical propositional logic is single premise, multiple conclusion. The step $p, q \vdash \phi \Rightarrow p \wedge q \vdash \phi$ is a formal way of recognising that *p* and *q* represent lattice points that may be combined into a single lattice point defining a filter that contains them both. It is customary to write the rules of logic in *multiple premise, multiple conclusion* form; this is not necessary. It is also customary to see expressions of the form $\Gamma \cup \{\phi\} \vdash \psi$ where Γ is a set of premises. This is either confusing or an error: sets represent disjunctive lists of members, for example, $\{1\} \cup \{2\} = \{1,2\}$, whereas in $\Gamma \cup \{\phi\} \vdash \psi$ the expression $\Gamma \cup \{\phi\}$ represents a conjunction of items in a list and corresponds to a meet in the lattice. So it is confusing to use the union symbol here, or to refer to Γ as a set; it is a conjunctive list not a disjunctive one.

10.4 (+) Definition, proof path, proof surface

Let $A \to B \to C \to \ldots$ be a chain of deductions, where each consequent is a valid inference from its antecedent. Let the propositions *A, B, C, ...* correspond to points in the lattice and the arrows be interpreted as directed paths joining those lattice points. The chain of deductions $A \to B \to C \to \ldots$ is said to be a *proof path*. Let Γ_A represent all valid proof paths starting at the point *A*. Let Π_A represent all paths starting with *A*, both valid and invalid – all paths leading away from *A* and joining *A* to any other point of space whatsoever. Then we call Γ_A a *proof surface*; it is a subspace of Π_A. The proof surface is also said to be the set of all *consequences* of *A*.

Assuming that *A* does not represent a tautology **1**, if the proof surface at Γ_A coincides with the space Π_A, then the logic is inconsistent. A logic can only be consistent if the proof surface at any given (contingent) point is a definite subspace of the entire space. Then there is a "force" that constrains all proof paths to lie on the proof surface. We ask, *what is the nature of this "force"?* In the case of analytic logic, that force arises from the actual analysis of some underlying space into its parts. In analytic logic the proof surface is a proper filter and the "force" is the rule that constrains all consequences of a proposition to lie in the filter it generates. But the notion of a proof surface is of more general application and need not apply only to lattices and their analytic logics.

Let γ, ψ be propositions corresponding to distinct lattice points; then $\gamma \vdash \psi$ means that there is a path in the lattice from γ to ψ. But this is ambiguous because if $\gamma \vDash \psi$ then *there must always be a path* from γ to ψ for that is also precisely what $\gamma \vDash \psi$ means. Therefore, *in this sense* (and only in this sense) we have, as a matter of definition, $\gamma \vDash \psi$ iff $\gamma \vdash \psi$. This property is known as *completeness*, so we may say that *in this sense* every logic is trivially complete by definition. This sense of completeness is trivial because *no* *distinction has been made* in it between the symbols $\vdash$ and $\vDash$. Then, to do so, we must recall that *logic is an application*, and for human beings *finite proofs* are essential. About this aspect Weyl wrote: -

> As I see it, mathematics owes its greatness precisely to the fact that in nearly all its theorems what is essentially *infinite* is given a finite resolution. But this "infinitude" of the mathematical problems springs from the very foundation of mathematics – namely, *the infinite sequence of the natural numbers and the concept of existence relevant to it.* "Fermat's last theorem," for example, is intrinsically meaningful and either true or false. But I cannot rule on its truth or falsity by employing a systematic procedure for sequentially inserting all numbers in both sides of Fermat's equation. Even though, viewed in this light, this task is infinite, it will be reduced to a finite one by the mathematical proof (which, of course, in this notorious case, still eludes us.). (Weyl [1994 / 1918] p.49)

The fundamental reason for distinguishing $\gamma \vDash \psi$ from $\gamma \vdash \psi$ is precisely because human reason seeks *finite proofs* of statements that are essentially *infinite* in conception. Up to now the only lattices we have considered have been finite. There is another meaning to *completeness* – namely, that the set of axioms and rules completely define the lattice in the sense that *if there exists a lattice path then a proof path can be given for it.* In this case the property of completeness is not trivial, since it is possible that the axioms and rules may not adequately characterise the underlying lattice. If the underlying lattice is finite, then, given any incomplete set of axioms and rules, it would be possible to extend it to a complete one; so we say that the logic is essentially complete. In this context, this essential completeness arises from the underlying finite nature of the lattices in question.

An infinite lattice may be incomplete *as a lattice.* A finite structure may behave in certain respects like a lattice and may be *lattice-like* without being a lattice because certain lattice points are missing. In this case, too, the structure may be said to be incomplete. If the underlying structure is incomplete then the logic built over it must inherit incompleteness from it. In conclusion: -

10.5 Definition, complete logic

1. A lattice is *incomplete* if not all unique joins and meets exist in the lattice. In that event, any logic built over such a lattice must also be incomplete.
2. A logic built over a lattice is *trivially complete* if the relation of proof path, $\gamma \vdash \psi$, is defined so that $\gamma \vDash \psi$ iff $\gamma \vdash \psi$.
3. A logic is *complete*, if given some set of axioms and rules of inference for the logic that define $\gamma \vdash \psi$, we have $\gamma \vDash \psi$ iff $\gamma \vdash \psi$.

To this we add the following definition: -

10.6 Definition, categorical logic

4. A logic is said to be *categorical*, if, given some set of axioms and rules of inference for the logic, there could be only one lattice up to isomorphism that matches it. Such a lattice is called a *model* for the logic. Another term for logic is *language*.

Categoricity implies some cardinality condition – in other words, definitely insists upon a lattice of a certain size; if the lattice has a skeleton, then the cardinality of that skeleton must also be defined. Discussion of categoricity inverts the general viewpoint of this paper – which is to start with lattices and investigate how those constrain analytic logics; the study of categoricity starts with a formal language and investigates the question: what models could match this language? The standard axiomatisation of the propositional calculus is not categorical in this sense, because any finite Boolean lattice is a model for it. The axioms are all true in the minimum (and only) indecomposable lattice $\mathbf{2} = \{0,1\}$, which is a factor of every (finite) Boolean lattice.

10.7 Intuitionism

By intuitionism here I refer to the formal theory of infinite valued logics wherein the law of excluded middle does not apply. So far I have not discussed this theory as a distinct alternative to classical logic. The device of approaching the subject from the lattice first makes this unnecessary. This is owing to firstly the observation that the underlying model of any intuitionist logic is a distributive lattice, and then to the following theorem: -

10.8 Theorem

Every distributive lattice can be embedded in a complete Boolean algebra.
(For proof, see Crawley and Dilworth [1973] p.89. Birkhoff [1940])

This theorem means that every observation on Boolean lattices and classical logic made in this paper with regard to the problem of formalism is automatically inherited by distributive lattices and intuitionist logic.

11 The two-in-one problem

There is a distinction of general lattice theory that is of vital importance to this evaluation of Poincaré's thesis. This is the distinction between a chain and an antichain.[20]

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[20] The technical details on which this section is built may be found in chapter 8 of Davey and Priestley [1980]. It is the material in this chapter that acts as the essential foundation of the refutation of formalism and its associated metaphysical thesis of strong AI.

11.1 Definition, antichain

An ordered set P is an *antichain* if $x \le y$ in P only if $x = y$. (Davey and Priestley [1990] p. 3)

A chain is a set of isolated points with no covering relations upon them. We begin with a partition of space, which is an antichain of atoms. This collection of atoms acts as a skeleton for the lattice constructed over it. Davey and Priestley comment, "Atoms were the right building blocks for finite Boolean algebras. In order to represent finite distributive lattices we need a more general notion – that of a join-irreducible element." (Davey and Priestley [1990] p.165)

11.2 Definition, join irreducible

An element x of a lattice L is said to be *join-irreducible* if: -

1. if L has a zero then $x \ne 0$
2. $x = a \vee b \Rightarrow x = a$ or $x = b$ for all $a, b \in L$

Condition (2) may be replaced by: -

2* $a < x$ and $b < x$ imply $a \vee b < x$ for all $a, b \in L$.

Meet-irreducibility is defined dually. The set of join irreducible elements is denoted $\mathbf{J}(L)$. The set of meet irreducible elements is denoted $\mathbf{M}(L)$

In a Boolean algebra the set of atoms is equal to the set of join irreducible elements. It is remarked, "In a finite lattice L, an element is join-irreducible if and only if it has exactly one lower cover. This makes $\mathbf{J}(L)$ extremely easy to identify from a diagram of L." (Davey and Priestley [1990] p.166) In a Boolean algebra every element, except the atoms, is a join of at least two other elements, so is not join-irreducible. However, in a general lattice we get elements that are lying one unit above another element and are not joins of two elements – these are join irreducible. At the opposite end of the spectrum to an antichain is a chain. [Chap. 2, Def. 2.5.3] A chain is a *totally ordered set.*

11.3 Join-irreducible elements of a chain

In a chain every non-zero element is join-irreducible. Hence if L is an n-element chain, then $\mathbf{J}(L)$ is a $n-1$ element chain.

It is observed that in a lattice L with no infinite chains, the ordered set $P = \mathbf{J}(L) \cup \mathbf{M}(L)$ meets the first two criteria for a skeleton, since the whole lattice may be described as the completion (by Dedekind-MacNille completion) of this set. (See Davey and Priestley [1980] chapter 8). I cite though will not prove here the Boolean (Birkhoff) representation theorem. [Proven in 7.2.9]

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1.4 Birkoff representation theorem

In a finite distributive lattice L we have the ordered set $P = \mathbf{J}(L)$. Then it follows that $L \cong \mathbf{O}(P)$ where $\mathbf{O}(P)$ is the family of all ideals of the lattice L.

This means that any distributive lattice is isomorphic to the family of all ideals, and the skeleton generates it as such. This will be clearer when we consider the Boolean Prime Ideal theorem in more detail. [See 7.2.9] The essential fact that we are seeking immediately in this context is the following.

11.5 Lemma

Let $L = \mathbf{O}(P)$ be a finite distributive lattice. Then

1. L is a Boolean lattice iff P is an antichain, in which case $\mathbf{O}(\bar{\mathbf{n}}) = \mathbf{2}^n$.
2. L is a chain iff P is a chain; $\mathbf{O}(\mathbf{n}) = \mathbf{n} + 1$.

(See Davey and Priestley [1980] p.173).

So we get a Boolean algebra if we have a skeleton of atoms, and something else otherwise. For complete induction we require the <u>chain</u> $\mathbb{N} = \{0,1,2,3, ...\}$, yet to construct the Boolean algebra upon it we must treat it as if it were a set of atoms – that is, as an antichain.

11.6 (+) The two-in-one problem

Herein lies the limitation of analytic logic built over a lattice. Logic in general is concerned simultaneously with two distinct lattice structures that *appear* to be incommensurable in one single lattice. The order relation of $\mathbb{N}$ is needed for complete induction; the unordered antichain of an infinite partition is required for analytic logic. I call this the *two-in-one* problem.

We shall learn that the two-in-one problem is solved in an actually infinite lattice, but the manner in which it is solved will demonstrate categorically the falsity of formalism and the validity of Poincaré's thesis.

CHAPTER 5

Infinite Lattices

1 The calculus of "predicates"

Up to now the subject of our enquiry has been the classical propositional calculus and its correspondent model, the class of all finite Boolean lattices: $\{B : B \cong \mathbf{2}^n, n \in \mathbb{N}\}$. The predicate calculus adds to the language of the propositional calculus: *names* of *individuals* belonging to some *domain* or *universe of discourse*; *variables standing for these names* (*ranging over the domain)*, *predicate symbols,* and *quantifiers*. In *first-order logic* there are also *function symbols*, but we concentrate for present just on the predicate part of the calculus. A typical formula of the predicate calculus is just Pa, where P stands for a predicate and a for an individual [Defined, chap. 2 / 2.1]; the standard example is, "Socrates is mortal" where a is an alternate name for Socrates ("Socrates" is a name of Socrates) and P is the predicate "... is mortal", which *might* from a philosophical point of view be said to denote the property of mortality[1]. Such a reading is an example of an application of predicate logic – and another instance of the attempt to "force" natural language into the confines of formal logic. In mathematical logic any pretension to be dealing *directly* with natural language is immediately dropped and we decide from the outset that our individuals shall be mathematical entities of some kind – numbers or sets. The predicates are number-theoretic or set-theoretic predicates – for example, "... is even" or "... is less than ...".

We can introduce such predicates into a finite language. For example, if the base set is given by $A = \{1,2,3,4\}$ where 1,2,3,4 now really are *numbers* and no longer mere partitions of space, then each subset of A shall define a predicate: $\{2,4\}$ is the predicate "... even number in A". Two predicates shall be regarded as being identical if they share the same extension. [Chap.2, Sec. 1.3.1] The symbol $\{1\}$ denotes the predicate "... is a member of A and is identical to 1." Thus, we see automatically that in this form the following result: -

1.1 (+) Result

In the formal analytic predicate calculus *all predicates can be eliminated in terms of propositions.*

[1] In this essay I am not concerned with the ontology of properties or properties, and neither assume them, nor discount them.

Proof

If $P \equiv$ "is a member of $\{1\}$" and $a = 1$, then Pa is the proposition $1 \in \{1\}$ and represents an atom, α_1, in the Boolean algebra defined over A, which is isomorphic to $\mathbf{2}^4$; that is $\mathbf{P}(A) \cong \mathbf{2}^4$.

Predicate calculus also embraces rules for the use of quantifiers; universal instantiation is illustrated by the rule: -

$$\frac{\vdash Pa}{\vdash (\forall x) Px}$$

But the quantifier is eliminable in favour of a list, $(\forall x) Px \equiv P1 \wedge P2 \wedge P3 \wedge P4 \wedge \ldots$. This rule reduces in the finite case of $\mathbf{P}(A) \cong \mathbf{2}^4$ to $(P1 \wedge P2 \wedge P3 \wedge P4) \vdash (P1 \wedge P2 \wedge P3 \wedge P4)$, which is a "mere tautology". The use of predicates in the finite calculus of predicates (a form of formal analytic logic) is nothing more than a *façon de parler*, or at best a tool of convenience, and that *there are no true predicates in this calculus.* Surprisingly, the same principle extends to the infinite case as well, provided that we allow for infinite lists and infinite meets and joins, which we do when we claim that an infinite lattice is complete. [Defined 4.9.4] Even if the lattice is not complete in the sense that every infinite collection of lattice points has both a join and meet, it may still allow for some infinite joins and meets. Thus, in the logic that is built over the lattice there may be quantifiers, but in the lattice there are no distinct points that correspond to quantifiers that do not represent points already existing in the lattice. Quantifiers serve to *distinguish*, that is mark out and identify, certain points in the lattice; they do not create them.

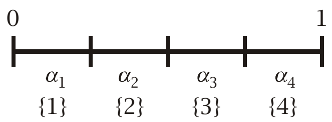
In scaling up from a finite to an infinite lattice we have stepped from a model of a lattice in which there finite meets and joins to one in which there exist at least some infinite meets and joins, if not *all* of them, as the lattice may be incomplete.

1.2 On the nature of a true predicate logic

In Aristotle's logic subject and predicate are said to be "terms", but the distinction between them is founded on *ontology* and not grammar: a subject is a term denoting an individual, and a predicate is a term denoting a universal. Thus, the Aristotlean subject/predicate distinction cannot be divorced from a theory of judgement, and all the mentalistic "baggage" that could attach itself to such a theory; it is a form of the logic of intensions not extensions. This constitutes a true predicate logic.[2] Modern formal predicate calculus is not a true predicate logic.

[2] For confirmation of this historical perspective consider the following observations made by G.H.R. Parkinson in his introduction to the logic of Leibniz: "He [Leibniz] states the subject/predicate distinction. He next proposes, as a task for inventive logic, the problem of determining all the possible predicates of any given subject, and all the possible subjects of any given predicate." A term is either a subject or predicate. "He is clearly using 'term' in the traditional sense of the subject or predicate of a proposition, and the fact that he speaks in this context of an alphabet of human *thoughts* indicates that he regards

Consider atoms of a finite Boolean lattice – taking $\mathbf{2}^4$ as our paradigm. These atoms arise from the conception of a division of space $[0,1]$ into disjoint mutually exclusive parts.



The atoms are propositions that name these partitions. In the predicate calculus our focus of interest has already shifted to natural numbers, sets and the continuum (points). The atoms are now *points in space*. Thus, *dimension* makes its appearance, and we require *variables* to distinguish coordinates in some space $\mathbb{N}^k$ or $\mathbb{R}^k$. The expression $x = y$ when it appears in algebra affirms that the first coordinate *x* is numerically equal to the second coordinate *y*. It is a statement about vectors. Thus, $x = y$, or perhaps more perspicuously, $x_1 = x_2$, makes its appearance when it is implicit that the domain is some 2- or *n*-dimensional space, and the individuals are pairs or *n*-tuples. That is why $x = y$ defines a *diagonal set* as a subspace of the universe of discourse, which is implicitly the Cartesian product $X \times X$ of some base set *X*. Likewise, $(\exists x)Px$ is related to a *projection function* from a space X^n onto its (first) coordinate axis. The atoms of $\mathbb{N}^2$ correspond to points of ordered pairs, and we can enumerate them: -

$(0,0), (0,1), (0,2), \ldots, (1,0), (1,1), (1,2), \ldots$

By diagonalisation these may be combined into a single list. Predicates may be represented as partitions of this $\mathbb{N}^2$ space (which is the domain of discourse), thus: -

$\{(0,0)\}, \{(0,1)\}, \{(0,2)\}, \ldots, \{(1,0)\}, \{(1,1)\}, \{(1,2)\}, \ldots$

The atomic propositions take the form: $(0,0) \in \{(0,0)\}$.[3] This can be abbreviated to just $\{(0,0)\}$, which also serves as the name of the atom. The predicate $x = y$ or $x = y \equiv \{(x,y) : x = y\}$ is the diagonal set, $\{(0,0), (1,1), (2,2), \ldots\}$ [See 2.5 below]. An example of

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such terms as concepts. The analysis, then, is one of concepts; stated roughly, Leibniz's view is that every concept is either ultimate and indefinable, or is composed of such concepts. The indefinable concepts are called by Leibniz 'first terms', and a list of these constitutes what he was later to call the 'alphabet of human thoughts', for derivative concepts are formed from first terms in much the same way as words are formed from the letters of the alphabet. Leibniz proposes to regard the first terms as constituting the first of a series of classes; the second class of the series consists of the first terms arranged in groups of two; the third class, of the first terms arranged in groups of three; and so on." (Parkinson [1966] xvii.) Parkinson goes on to state that in the *De Arte Combinatoria* Leibniz "follows Hobbes by regarding predication in terms of addition or subtraction, or he follows Aristotle and scholastic tradition by speaking of the mind 'compounding and dividing'." The identity theory is a more 'mature' concept where "Leibniz's view that to assert a proposition is to say that once *concept* is included in another – that is, his 'intensional' view of the proposition." (Parkinson [1966] xiv.)

[3] Atoms are asserted contingently, so the expression $(0,0) \in \{(0,0)\}$ means $(0,0)$ is contingently given: or "Consider a model in which $(0,0)$ is given."

a relation is, $R(x,y) \equiv \{(x,y): y = x^2\} = \{(0,0),(1,1),(2,4),...\}$. The predicate, $Py \equiv (\exists x)(y = x^2)$ enumerates the image set of this relation: $Py \equiv (\exists x)(y = x^2) = \{0,1,4,9,16,...\}$, so it is the projection of the relation *R* onto the *y*-axis. [See 2.2 below]

1.3 Quantifiers

Now we have to interpret quantifiers. For a finite join, $(\exists x)Px \equiv P_1 \vee P_2 \vee ... \vee P_n$, corresponds to a finite set. In an infinite lattice a predicate may correspond to an infinite set. No predicate (join) is necessary, except **1**. The impossible predicate is **0**.

Existential quantifiers pick out points in a lattice, given contingently, and hence also their filters. The "natural" scope of the existential quantifier in the arbitrary statement $(\exists x)Px$ is that which is may be contingently asserted (possibility). By contrast the scope of the universal quantifier in $(\forall x)Px$ is that which is necessary. If $(\forall x)Px$ is true, then *P* is a necessary predicate whose scope is the whole set of atoms (the base set). In other words $\vdash (\forall x)Px$ is a synonym of $\vdash (\forall x)x = x$ and is a name of **1**. Strictly, $\vdash (\forall x)Px$ cannot be asserted contingently; if it is asserted at all, it must be asserted as a necessary proposition, that is, as a name of **1**. $(\exists x)x \neq x$ is necessarily false, and is a name of **0**.

1.4 Rule for generalization

The rule for the introduction of the universal quantifier[4] is: -

$$\frac{\vdash \phi}{\vdash (\forall x)\phi x}$$

where ϕ is any well-formed formula of the logic.

This rule is partly responsible for the illusion that analytic logic is *not vacuous*; it appears to allow for the deduction of a universal generality $(\forall x)\phi x$ from finite information. One only has to examine the rule to see that this must be a false impression. The statement $\vdash \phi$ could only be true if ϕ was a name of **1**, since it is affirmed categorically, that is, without premise. So $(\forall x)\phi x \equiv \mathbf{1}$, and the inference reduces to $\mathbf{1} \vdash \mathbf{1}$. In practice, "substantive" uses of generalization appear in results such as, $(\forall x)(Fx \wedge Gx) \dashv\vdash (\forall x)(Fx) \wedge (\forall x)(Gx)$, and are "useful" in the manner in which tautologies in general are useful. Strictly speaking, there is no contingent meaning to $\vdash (\forall x)Px$ and all instances of this formula are disguised names of **1**.

1.5 Example

By the principle of dilution, the inference $\vdash (\forall x)Px \equiv \vdash (\forall x)(Fx \supset Gx)$ should be interpreted as: -

[4] For example, see Mendelson [1979] p. 60.

$$
\begin{aligned}
& \vdash (\forall x)(Fx \supset Gx) \\
\Leftrightarrow\quad & \vdash (\forall x)(P_1x \supset P_1x \vee P_2x \vee ...) \\
\Leftrightarrow\quad & \vdash (\forall x)(\neg P_1x \vee P_1x \vee P_2x \vee ...) \\
\Leftrightarrow\quad & \vdash (\forall x)(\neg P_1x \vee P_1x) \\
\Leftrightarrow\quad & \vdash (\forall x)\mathbf{1} \\
\Leftrightarrow\quad & \vdash \mathbf{1}
\end{aligned}
$$

Here also I will not allow $(\forall x)\phi x$ unless it quantifies over an infinite domain. Then: -

$(\forall x)Px \equiv Pa_1 \wedge Pa_2 \wedge Pa_3 \wedge ...$

where the list on the right-hand side is infinite.

1.6 Lattice inferences involving quantifiers

It would be useful to interpret in terms of the lattice the meaning of the valid inferences: -

$(\forall x)(Fx \wedge Gx) \dashv\vdash (\forall x)(Fx) \wedge (\forall x)(Gx)$

$(\forall x)(Fx) \vee (\forall x)(Gx) \vdash (\forall x)(Fx \vee Gx)$ But not conversely

$(\exists x)(Fx \vee Gx) \dashv\vdash (\exists x)(Fx) \vee (\exists x)(Gx)$

$(\exists x)(Fx \wedge Gx) \vdash (\exists x)(Fx) \wedge (\exists x)(Gx)$ But not conversely

Firstly I observe that all the inferences involving only the universal quantifier are strictly *pseudo-inferences lying in the pseudo-lattice* of **1**. This follows from the observation that $(\forall x)Fx$ is strictly a *name* of **1**, so any joins and meets of propositions of this type belong *not* to the lattice of joins (and meets) of atoms, but to the *pseudo-lattice* of **1**. The inferences

$(\forall x)(Fx \wedge Gx) \dashv\vdash (\forall x)(Fx) \wedge (\forall x)(Gx)$

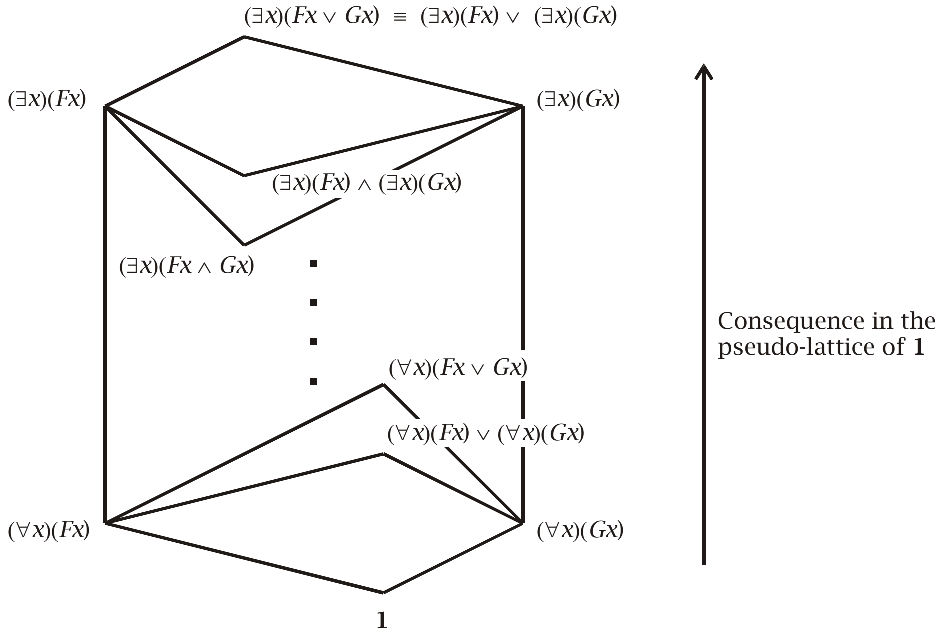
$(\forall x)(Fx) \vee (\forall x)(Gx) \vdash (\forall x)(Fx \vee Gx)$ But not conversely

belong to the pseudo-lattice, whereas the inferences

$(\exists x)(Fx \vee Gx) \dashv\vdash (\exists x)(Fx) \vee (\exists x)(Gx)$

$(\exists x)(Fx \wedge Gx) \vdash (\exists x)(Fx) \wedge (\exists x)(Gx)$ But not conversely

belong to the proper lattice.

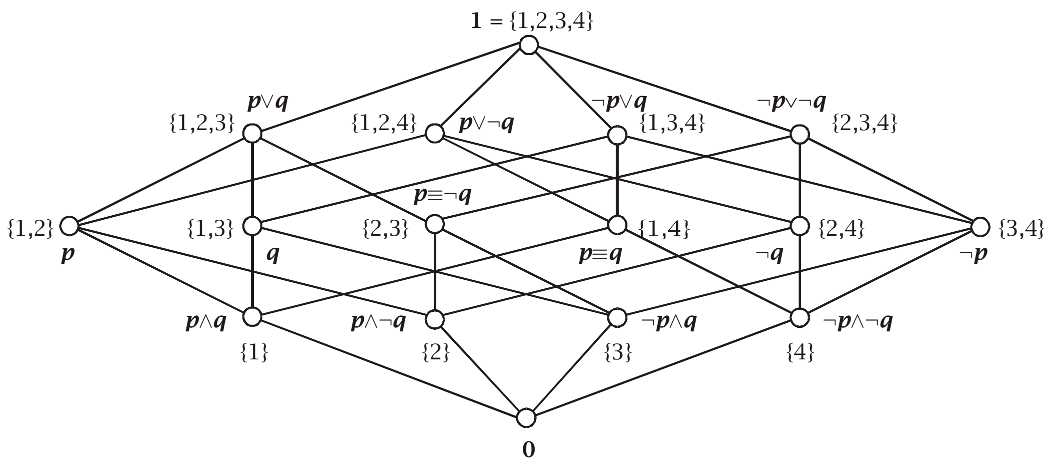


1.7 Logic of “identity” - Equations

The introduction of = into the predicate calculus concerns relations on the ideals of the lattice. Each ideal corresponds to an equation. First, an illustration: -

1.8 Example

In the lattice $\mathbf{2}^4$: -



let

$p \equiv x = y \equiv t \neq 3,4 \leftrightarrow \{1,2\}$, $q \equiv y = z \equiv t \neq 2,4 \leftrightarrow \{1,3\}$ and $x = z \equiv t \neq 2,3 \leftrightarrow \{1,4\}$.

Then

$p \wedge q \equiv (x = y) \wedge (y = z) \equiv t \neq 2,3,4 \leftrightarrow \{1\}$ and $(x = y) \wedge (y = z) \vdash x = z$ corresponds to the inference $p \wedge q \vdash p \equiv q$. This is equivalent to the relation that the filter generated by the node $p \equiv q$ is contained as a subset in the filter generated by the node $p \wedge q$.

1.9 (+) Proposition

Equation logic is the logic of filters.

Proof

In disjunctive normal form a node in a lattice is a disjunct of propositions. Assume that to each node *p* there is a conjunction

$$p \equiv \alpha_{i_1} \wedge \ldots \wedge a_{i_k}$$

where *k* is an ordinal and each α_i is an atom. We need to show that there is a relation on the set of all filters that is an equivalence relation and hence justifies the introduction of the equality symbol into the logic. Reflexivity and symmetry will be the easy case. It is the transitive relation that is needed: $x = y \wedge y = z \Rightarrow x = z$. Putting

$$p \equiv x = y \equiv \alpha_{i_1} \wedge \ldots \wedge a_{i_\lambda} \equiv (t \neq a_{i_1}) \wedge \ldots \wedge (t \neq a_{i_\lambda})$$
$$q \equiv y = z \equiv \alpha_{j_1} \wedge \ldots \wedge a_{j_\mu} \equiv (t \neq a_{j_1}) \wedge \ldots \wedge (t \neq a_{j_\mu})$$

Then $p \wedge q \equiv (t \neq a_{i_1}) \wedge \ldots \wedge (t \neq a_{i_\lambda}) \wedge (t \neq a_{j_1}) \wedge \ldots \wedge (t \neq a_{j_\mu})$. The statement $x = z$ will be equivalent to a sub-conjunction of this list. To an equation $x = y$ there corresponds a subset of the power set of the universe of discourse, which also corresponds to a node in the lattice. Denote the filter generated by this node by $F_{x=y}$. We have, for example: -

Equation		Partition	Filter
$x = y$	$\longleftrightarrow$	$\alpha(x = y)$	$F(\alpha(x = y)) = \{\beta \in P(V) : \alpha \subseteq \beta\}$
p		$= \{1,2\}$	$\{\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}\}$

The general relation on which equation logic is based is: -

$$F(\alpha(x = z)) \subseteq F(\alpha(x = y) \cap \alpha(y = z))$$

The inclusion is strict if $x \neq y$. In the preceding above ($\mathbf{2}^4$) we have $x = z \leftrightarrow p \equiv q$ - this is based on the universe $V = \{1,2,3,4\}$. In a larger universe the statement $x = z$ is equivalent to an unspecified disjunctive normal form. Introduction of = into the language creates a distinguished point in the lattice corresponding to a filter.

Thus, axioms governing the = symbol are based upon an existing relation in the lattice. The lattice induces a relation on filters, and the = symbol creates a convenient language to describe this relation. It maps back that relation to a relation between lattice points. The relation between lattice points *appears* to be not a law of the lattice, but this is an appearance only. To each lattice relation there corresponds an algebraic lattice law, though the correspondence is a function of the size of the universe and the interpretation of the equation in that universe: -

Lattice		Filters
$p \equiv x = y$	$\xrightarrow{\text{induces}}$	$F_{x=y}$
		$\downarrow$
$x = y \wedge y = x \vdash x = z$	$\xleftarrow[\text{pull back}]{}$	$F_{x=z} \subseteq F_{x=y} \cap F_{y=z}$

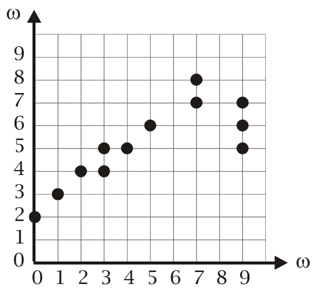
The axioms governing = do not correspond to points in the lattice – they are distinguished elements from the set of logical laws. There is only one way to add the axioms. They are not contingent structures but alternative descriptions of the structure of the lattice. Adoption of the axioms of = are *forced* by the axioms of the lattice. It is the only consistent way to extend the lattice to include in the language the = sign. The relation already exists and all that is supplied is a name.

2 Binary and *n*-ary relations within predicate logic

When we deal with binary relations the atoms of the partition are treated as ordered pairs.

2.1 Binary relations

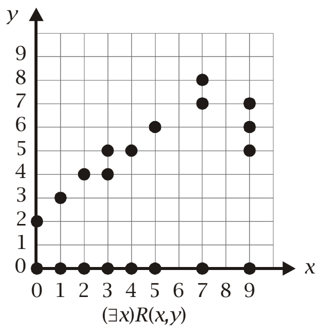
A binary relation is a subset of points from the set of ordered pairs $\mathbb{N} \times \mathbb{N}$.



Example of a binary relation

$R(x,y) = \{(0,2),(1,3),(2,4),(3,4),(3,5),\ldots\}$

2.2 Existential quantifier over a relation



Example of an existential quantifier over a relation.

$(\exists x)R(x,y) = \{x \in \omega : R(x,y)\} = \{0,1,2,3,4,5,7,9,\ldots\}$

The existential quantifier is equivalent to the projection of the binary relation $R(x,y)$ onto the *x*-axis, or 1st coordinate. Likewise, for $(\exists y)R(x,y)$ this is the projection onto the 2nd coordinate.

2.3 Negation and universal quantifier

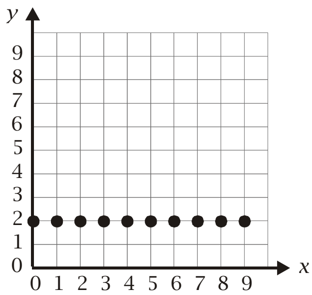
$\neg R(x,y) = \{(0,0),(0,1),(0,3),\ldots(1,0),(1,1),\ldots\}$

represents all those points in $\mathbb{N}\times\mathbb{N}$ not in *R*.

$(\exists x)\neg R(x,y) = \{0,1,2,3,4,\ldots\} = \mathbb{N}$

$(\forall x)R(x,y) = \neg(\exists x)\neg R(x,y) = \varnothing$

2.4 Constant function



$f(x,y) = \{(0,2),(1,2),(2,2),\ldots\}$

$(\exists x)f(x,y) = \{0,1,2,3,\ldots\}$, true

$\neg f(x,y) = \begin{Bmatrix}(0,0),(0,1),(0,3),\ldots, \\ (1,0),(1,1),(1,3),\ldots,(3,0),(3,1),\ldots\end{Bmatrix}$

$(\exists x)\neg f(x,y) = \{0,1,2,3,4,5,\ldots\}$, true

$(\forall x)f(x,y) = \neg(\exists x)\neg f(x,y) = \varnothing$, false

$(\exists y)f(x,y) = \{2\}$, true

$(\exists y)\neg f(x,y) = \{0,1,3,4,5,\ldots\}$, true

$(\forall y)f(x,y) = \neg(\exists y)\neg f(x,y) = \{2\}$, false

$(\forall y)(\exists x)f(x,y) = \varnothing$, false

$(\exists x)(\forall y)f(x,y) = \varnothing$, false

$(\exists x)(\exists y)f(x,y) = 1$, true

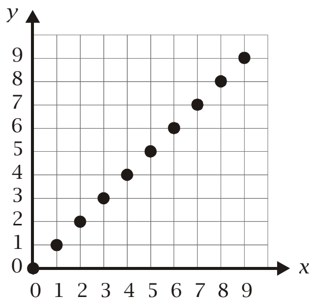
$(\forall y)f(x,y) = \neg(\exists y)\neg f(x,y) = \{2\}$, false

When we get down to fully bound relations, for example,

$(\forall y)(\exists x)f(x,y)$, $(\exists x)(\forall y)f(x,y)$, $(\exists x)(\exists y)f(x,y)\ldots$

the only possible values are 0, 1 indicating whether the relation holds or not; whether the expression is true or false. A statement with a universal quantifier $(\forall x)f(x,y)$ can only be true if the projection set is the entire set: $(\forall x)f(x,y)$ is true $\Leftrightarrow (\forall x)f(x,y) = 1$.

2.5 Diagonal relation



$D(x,y) \equiv (x = y) = \{(0,0),(1,1),(2,2),\ldots\}$

$(\exists x)D(x,y) \equiv (\exists x)(x = y) = \{0,1,2,\ldots\} = \omega$, tr

$(\exists y)D(x,y) \equiv (\exists x)(x = y) = \{0,1,2,\ldots\} = \omega$, tr

$\neg D(x,y) \equiv (x \neq y) = \{(0,1),(0,2),\ldots(1,0),(1,2$

$(\exists x)\neg D(x,y) \equiv (x \neq y) = \{0,1,2,\ldots\} = \omega$, true

$(\forall x)D(x,y) = \neg(\exists x)\neg D(x,y) = \varnothing$, false

$(\exists y)(\exists x)D(x,y) = 1$, true

$\neg(\exists x)D(x,y) = \varnothing$, false

$(\exists y)\neg(\exists x)D(x,y) = \varnothing$, false

2.6 All relation

All points in the *x,y*-plane.

$(\exists x)A(x,y)=\{0,1,2,...\}=\omega$, true

$\neg A(x,y)=\varnothing$

$(\exists x)\neg A(x,y)=\varnothing$, false

$(\forall x)A(x,y)=1$, true

$(\forall y)(\forall x)A(x,y)=1$, true

The universal quantifier has only two values – 1 = true and 0 = false. There is a lack of symmetry between the two quantifiers. The existential quantifier picks out a set within the domain, equivalent to a disjunction. The universal quantifier denotes a valuation that is not contained within the lattice.

2.7 Empty relation

$R(x,y)=\varnothing$

The only relation for which we have

$(\exists x)R(x,y)=\varnothing$, false

2.8 *n*-ary relations

n-ary relations are treated as product spaces, just as binary relation is a product space $\omega\times\omega$; the *n*-ary relation has *n* copies of the base set; the elements are *n*-tuples; the existential quantifier projects onto the *i*th coordinate; the members of the product spaces are *n*-ary relations which are pointsets. The largest space is Baire space – the set of all *n*-ary tuples, $\mathcal{N}=\omega^{\omega}$.

3 The actually infinite partition of the interval

The set $\mathbb{N}=\{0,1,2,3...\}$ represents the set of all natural numbers. This is a potentially infinite set, for we can never have done with enumerating all the natural numbers; as a collection it is inexhaustible. Cantor allowed the set $\omega=\{0,1,2,3..\}$ to be the actually infinite collection of all ordinals. This is in accordance with the usual definition of an ordinal number that one encounters in set theory: Let *n* be an ordinal; then the ordinal successor of *n* is $n'=n\cup\{n\}$.[5] From this definition it follows for an ordinal, $n=\{0,1,2,...,n-1\}$; thus, an ordinal is a set that contains every ordinal that precedes it in the series of all ordinals. It seems natural to extend

[5] One could introduce separate notation for natural numbers and ordinals, but I do not think that would make the exposition here any clearer, and I rely on context to indicate which of the two is intended.

this definition to the infinite case, so that we have ω to be the next ordinal that comes after all the finite ordinals though not as the successor of any of these; it is the least infinite ordinal. It is a natural enough definition and fits well with the purpose that Cantor originally intended it for.[6]

3.1 The paralogism of formalism

The Cantorian approach is the occasion for the paralogism that underpins formalism. [Defined, Chapter 1, Section 1 et seq.]

1. Arithmetic is based on the collection $\mathbb{N} = \{0,1,2,3...\}$, which is a potentially infinite collection of natural numbers that have no upper bound. This is equivalent to the **Archimedean property**[7] that the natural numbers are not bounded above. $\mathbb{N}$ is a collection equipped with a *synthetic* principle of reasoning known as *complete induction*, by means of which a conclusion about the *entire collection*, $\mathbb{N}$, is attained. Since this principle makes a conclusion about the whole, by this means the *finite has been decidedly transcended*; the *infinite is reduced to the finite*. Our conclusion applies to *all of* $\mathbb{N}$, not just a part. In this sense, it is a conclusion about a "completed" infinity. However, the completion involved in this process of inference is in no wise the same actual collection conceptualised in $\omega = \{0,1,2,3...\}$; we never in complete induction conceive of the totality of all the natural numbers, or imagine that these numbers *have a successor*. So ω is foreign to both $\mathbb{N}$ and complete induction.
2. Analysis is based on the continuum, which is primarily the notion given to intuition of a continuous extension capable of being analysed into parts. A division of the continuum into a finite number of discrete parts is an insufficient basis for science; it is *incomplete*. Therefore, an actually infinite division of the continuum is required.[8] (It also being the case that a potentially infinite division of the continuum is insufficient.) [See 3.7 et seq. below.] This actual infinite partition of the continuum gives rise to the notion of ω as the actually infinite collection of all ordinal numbers;

[6] An analysis of the properties of the continuum with a view to identifying just when such and such a trigonometric series converges and under what conditions. See Dauben [1979] and Bessoud [2008].

[7] **Archimedean property**: If a and b are any particular integers, then there exists a positive integer n such that $na \geq b$. (Burton [1976] p.2) This implies that $\mathbb{N}$ is not bounded above.

[8] This necessarily involves the notion of an actual infinity of parts and gives rise to certain tensions that manifest themselves as paradoxes, or rather problems. The division may be thought of as taking place through an actual process of dividing the line and each division may be counted: 1,2,3, ...; however, it becomes clear to the understanding that a process that increases in tandem with the counting numbers is still insufficient to provide a complete system of explanation of the parts of the real line. It is necessary for a *complete* understanding of the continuum sufficient to serve as a basis for science to attain to the notion of a *real number* corresponding to a *point of the continuum* . To arrive at the collection of all real numbers we must conceive of a process of division that is *more rapid* than any process of counting by means of finite numbers could be. From this arises the notion of *different sizes of infinity*.

a limit ordinal is defined to be a non-zero ordinal which is not the successor of any other ordinal; ω is defined to be the least limit ordinal. The difference between $\mathbb{N}$ and ω is encapsulated by this theorem: -

3.2 Lemma

For ω the following statements are equivalent: -

1. ω is a limit ordinal
2. $(\forall n)(n < \omega \supset n+1 < \omega)$
3. $\omega = \sup_{n<\omega} n$

(Proof in Potter [2004] p. 181)[9]

Thus ω is the least upper bound (supremum) of all natural numbers. Thus, if $\omega = \mathbb{N}$ we must allow the collection of all natural numbers to be bounded, contradicting the Archimedean property. We must either dispense with the Archimedean property or drop the identification $\omega = \mathbb{N}$. The error of identifying $\mathbb{N}$ with ω is common: -

> ... there is a least limit ordinal, which is called ω ("omega"). The members of ω are called **finite** ordinals or **natural numbers**. In other words, to a set theorist $\omega = \mathbb{N}$. (Wolf [2005] p. 83) [See also Chapter 1, section 4 for other instances.]

Not only is $\omega = \mathbb{N}$ a mistake, but the paralogism of formalism arises as follows: analysis of the continuum into actually infinite parts enables the creation of an analytic logic of the continuum. Within this logic it is possible to construct a *theory of chains* and within this theory it is possible to embed an *analytic variant of the principle of complete induction.* The paralogism is to take this analytic variant for the original principle of complete induction, which then makes it *appear* that arithmetic is a version of analytic logic. This is an illusion.

I shall demonstrate that $\omega = \mathbb{N}$ leads to a contradiction in the theory of cardinal numbers in Chapter 6. [Chap.6 /.3.9] This will demonstrate that set theory requires $\omega \neq \mathbb{N}$.

The Archimedean property is a synthetic principle of the natural numbers in its own right. But it also a deductive consequence the Completeness axiom, and hence, if we eliminate the Archimedean property we must drop the Completeness axiom. This is shown as follows: -

3.3 Analytic proof of the Archimedean property from the Completeness Axiom

Suppose $\mathbb{N}$ is bounded above. Then by the completeness axiom there exists a unique real number *u*, such that $u = \sup \mathbb{N}$. For any number $n \in \mathbb{N}$ the number $n+1 \in \mathbb{N}$, hence $n+1 \leq u$ and $n \leq u-1$. This is true for all $n \in \mathbb{N}$, hence $u-1$ is an upper bound for $\mathbb{N}$. This contradicts the uniqueness of *u*, so $\mathbb{N}$ cannot be bounded above.

---

[9] I have slightly adapted the theorem in Potter which is for all limit ordinals. Only ω concerns us here.

Hence, we see that the Completeness axiom requires $\omega \neq \mathbb{N}$. That this is entirely correct shall become transparent when we consider the actual partition of the interval into ω parts.

Furthermore, when we see that the categories of the *potential* and *actual* infinite are two entirely distinct categories of explanation and have origins that are distinct, we see that in trying to subordinate the one to the other we have made a fundamental category error. The potential infinite, as a separate category, has re-emerged even within set theory in the notion of a *proper class* – an infinite collection of sets *that is not a set* and *is not bounded above.*[10] Here the paralogism takes the form: (a) there is no model of set theory that is a set yet (b) all mathematics is set theory.

One only has to examine the definition of the ordinal ω to see that it does not encompass the potential infinite: ω *is a limit* and belongs to analysis not to arithmetic. In set theory we must *add an axiom of infinity*, that is, an axiom that permits the formation of a completed totality of actually infinite members; then, and only then, are we able to derive a species of complete induction. As a basis for the analysis of the continuum this way of proceeding is appropriate; but if it gives rise to the paralogism that we have *dispensed with synthetic reasoning altogether* then the damage done far outweighs the gains. The principle of complete induction is not the same as *taking a limit.*[11]

This conclusion is also supported by the following observation: when we take a limit we never do attain the limit; the process of taking a limit is a *façon de parler* for some constructive process that enables us to approximate a number that we could never truly construct as a thing-in-itself, though we might be able to name it. In classical, as opposed to constructive mathematics, we allow the notion of a limit to acquire a kind of theoretical supra-existence, but its relation to a constructive process is never wholly lost. On the other hand, in complete induction, *we really do attain the bound implied by the reduction of the infinite to the finite that it encompasses.* In an induction we truly pass from *finite information* to information about an infinite class of objects; from *any* to *all.* So, while induction never gives us the notion of an actual infinity (which does not belong to it as a category) it does give us knowledge of infinity. In precisely this way infinity is a concept that we do understand and grasp, albeit not in the same way that we grasp anything that is finite.

---

[10] I am aware of the claim that proper classes can always be eradicated formally in favour of sets, or definite multiplicities. I claim that this device fails to eradicate the *concept*; whatever the technical successes of such an approach (they are exaggerated) the concept is vital to our system of explanation. The concept has not been eradicated, and if it were, why would *proper class* appear as a term in any (substantial) textbook of set theory whatsoever?

[11] They are conceptually related, for both are connected to the synthetic grasp that an act that is repeated indefinitely gives rise to the concept of a totality of *all* repetitions of that act. But they are also conceptually distinct. In induction the potential is completed as an inference from *any* to *all*, but not in the sense of *adding a new number as the successor to all natural numbers.* In the taking of a limit, because of the synthetic connection to the continuum, the limit is conceived as being *one member of a completed, actual infinite totally of members.*

In a process of enumeration *infinity is never given*, and it is another aspect of the fundamental paralogism of formalism to take enumeration for infinity, either potential or actual. I can never tell from a mere sequence of numbers that knowledge about the entire sequence or a potentially infinite collection is encompassed by it. From 2, 4, 6, 8, ... what follows? The dots convey nothing unless I have already grasped the concept of their continuation in accordance with a rule, and without that rule no examination of the sequence will ever produce it. It is also insufficient to distinguish between whether the potential or actual infinity is intended. It is a fundamental error to presume that enumeration of lists is sufficient to explain anything.

3.4 Infinite partitions

The set of natural numbers, $\mathbb{N}$, is unbounded and the concept of infinity encompassed by this set is rightly denoted ∞. The set of finite ordinal numbers may be regarded as bounded above by means of a formal definition, and this upper bound is denoted ω, which is defined to be a *limit ordinal* and *that ordinal which follows all the finite ordinals in the succession of all ordinals*. By this means we encompass the idea of an actually completed totality of all finite ordinals and hence of all natural numbers that may be placed in correspondence with those finite ordinals. The structure, ω, owes its origin to the analysis of the continuum by means of an infinite partition. The continuum may be analysed in at least two different ways: -

1. By division into actually ω parts.
2. By division into actually more than ω parts.

By this second division I encompass the division of the real line into points and the correlation of those points with real numbers.[12]

My purpose here is to explore the analytic logic of the division of the real line into ω parts. Such a division could never suffice to produce a continuum; it produces a scaffold, or *skeleton* [Chap. 4, Sec. 11.3], of points of the continuum rather than the continuum itself. Upon this skeleton a Boolean lattice can be constructed. This is the structure $\mathbf{P}(\omega) \equiv \mathbf{2}^{\omega} \cong \{0,1\}^{\omega}$, which is known as the *Cantor set*. Any structure isomorphic to the Cantor set is known as a *Cantor space*. Since all Cantor spaces are identical up to isomorphism, it is possible to talk of only one Cantor space. Cantor space is the Boolean lattice/algebra of which ω is the skeleton. It can be shown that it is a complete, atomic lattice.

---

[12] Following Cantor's it has been customary to equate this number of parts with the power set of ω, denoted $\mathbf{P}(\omega)$. Each part, called a *point*, is said to correlate to a decimal number with infinite digits, which may or may not repeat. The cardinality of the continuum is denoted *c*. It is a theorem that $c = 2^{\aleph_0}$ where $|\omega| = \aleph_0$. I regard it as an open question as to whether such a division is sufficient to embrace or define the concept of continuity; in a formal sense it may do, but there are unresolved tensions involved with this definition

3.5 About zero

I shall include 0 in $\mathbb{N}$ by definition; that is, $\mathbb{N} = \{0,1,2,3,...\}$ where 0,1,2,3, ... are natural numbers.

Formalists must deny that the succession of natural numbers, 0,1,2,3, ... constitutes a primary category of understanding, and claim that all that can be known them is encompassed by the theory of ordinals and their position in a progression that could be mechanically generated. It is usual to adopt the von Neumann definition of ordinals: -

$$0 = \varnothing$$
$$1 = \varnothing \cup \{\varnothing\} = \{0\}$$
$$2 = 1 \cup \{\varnothing\} = \{0,1\}$$
$$...$$
$$n = \{0,1,\ ...\ n-1\}$$

This can create difficulties when discussing the Boolean lattice constructed over ω, for $0 = \varnothing$, denoting the empty set, is also the zero of the lattice. I denote the zero of the lattice by **0**, that is, using bold face type. The division of the line produces parts that correspond to singleton sets: $\{0\},\{1\},\{2\},...$; by the definition of the ordinal 1, we see that $1 = \{0\} = \{\varnothing\}$ represents one of these partitions. The ordinals in general do not stand for these partitions. The ordinals, 0,1,2,3,... in general represent the "content" of each partition, as in $1 \in \{1\}$, whereas the singleton set itself represents the partition. The partition may also be represented in other ways, for example, by singleton sets, $\{a\},\{b\},\{c\},...$ or by singletons containing successive primes: $\{2\},\{3\},\{5\},...$. Although the intention is to build a theory of arithmetic within the analytic logic $\mathbf{2}^{\omega} \cong \{0,1\}^{\omega}$ it should be noted at the outset that the concept of order is not instrumental in the construction of this lattice, so if the lattice embraces the order of inherent in the succession of natural numbers this must emerge in some other way.

3.6 (+) Summary

$\mathbb{N} = \{0,1,2,...\}$ is unbounded above. All sets are "determinate multiplicities" [Chap.2, Sec.1.3.1]. They are determined by their extensions. Although it is common to treat $\mathbb{N}$ as a set, it is not strictly a determinate multiplicity. It does not appear in set theory. It is customary to equate $\mathbb{N} = \omega$ - this is <u>a mistake</u>.

1.1 $\mathbb{N}$ is the unbounded <u>collection</u> of all natural numbers; ω is the least infinite <u>ordinal</u>, which is a set.

1.2 $\mathbb{N}$ is potentially infinite; ω is actually infinite.

1.3 Every element of $\mathbb{N}$ is unbounded above; every element of ω is bounded above, by ω, which is placed in succession after all these elements.

Despite the common mistake, illustrated by the quotation from Wolf above, of equating $\omega = \mathbb{N}$, some texts of set theory *make no reference whatsoever to* $\mathbb{N}$. For example, $\mathbb{N}$ makes *no appearance* in Levy [2002].[13]

3.7 One-point compactification

It is at this point, when we must consider how to divide the interval $[0,1]$ into an actually infinite number of partitions equinumerous to ω, that the tangible distinction between the potentially infinite $\mathbb{N}$ and the actually infinite ω makes a difference of momentous significance. The collection $\mathbb{N}$ is literally incapable of dividing the interval $[0,1]$ into an infinite number of segments for the reason that it represents a potential infinity. If we start numbering the partitions we will never have done, because $\mathbb{N}$ is unbounded above. $\mathbb{N}$ is a locally compact [Chap.2/2.9.6] but not globally compact set [Chap.2/2.9.5]. $[0,1]$ is bounded. A partition of $[0,1]$ by $\mathbb{N}$ segments is impossible because we are trying to divide the unbounded into the bounded.

In case the reader concludes that I am inventing some new issue and embarking upon a paralogism of my own, let me immediately state that this observation is actually *standard theory*, since it is universally allowed that the partition of $[0,1]$ requires a 1-point compactification. The half-open interval $[0,1)$ is locally compact but not closed or bounded above. In order to close it, we need to adjoin to it the neighbourhood of the point 1, here represented by $\{1\}$. That is, we write, $[0,1]=[0,1)\cup\{1\}$. From this there follows the Heine-Borel theorem that the interval $[0,1]$ is compact. The Heine-Borel theorem is, in turn, just one of many equivalent formulations of the Completeness Axiom [Chap.2 / Sec. 2.10]. That is to say, the very possibility of compactifying the half-open interval $[0,1)$ is equivalent to completing it and rests on a basis that, but for formalism, one would immediately conclude was a synthetic act of imagination. The interval $[0,1]$ is the sub-manifold upon which we are currently attempting to define a scaffold (skeleton) or partition of actually ω parts. A subdivision of $[0,1)$ would correspond nicely to a partition by $\mathbb{N}$ parts precisely because the sub-manifold is open and unbounded just as $\mathbb{N}$ is unbounded. Thus we can clearly "attach" $\mathbb{N}$ to $[0,1)$ using the famous method of Zeno in the paradox of the division of the line.[14] That

[13] There is no reference to $\mathbb{N}$ in Levy's index of notation. (Levy [2002] p. 378).

[14] For a description of Zeno's paradox of the Dichotomy and Aristotle's solution to this, see Barnes [1979] Chapter XIII.

is, by halving and having the interval $[0,1)$ *ad infinitum*. So manifestly $\mathbb{N}$ is insufficient to partition $[0,1]$. These are well-known results: -

3.8 Result

Every discrete space is locally compact, but not compact if infinite. (Bourbaki [1989a] p. 90)

3.6 Corollary

$\mathbb{N}$ is locally compact but not compact.

The *standard partition* may be found in any appropriate text. I take it from Davey and Priestley [1990] (p.197): -

3.9 One-point compactification of a countably discrete space

Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

Let $U \subseteq \mathbb{N}_\infty$.

Let *T* be the topology on $\mathbb{N}_\infty$ given by

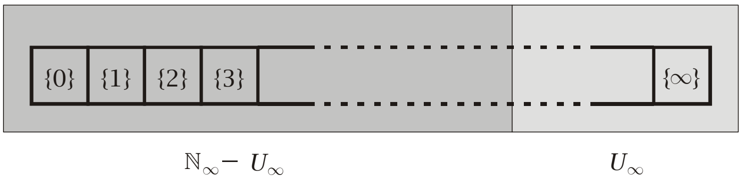
$$U \in T \text{ if } \begin{cases} \infty \notin U \\ \infty \in U \text{ and } \mathbb{N}_\infty - U \text{ is finite} \end{cases}$$

This can be shown to be a topology. (See Givant and Halmos [2009]). A subset $V \subseteq \mathbb{N}_\infty$ is clopen (both closed and open) iff *V* and $\mathbb{N}_\infty - V$ are in *T*. The clopen sets of $\mathbb{N}_\infty$ are the finite sets not containing ∞ and their complements. It can be shown that $\mathbb{N}_\infty$ is totally disconnected.[15]

---

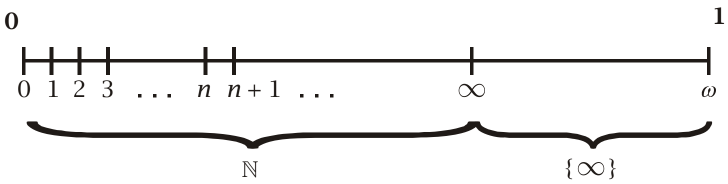
[15] Givant and Halmos write: "A less trivial collection of examples consists of the one-point compactifications of infinite discrete spaces. Explicitly, suppose a set *X* with a distinguished point x_0 is topologized as follows: a subset of *X* that does not contain the point $\{x_0\}$ is always open, and a subset that contains x_0 is open if and only if it is cofinite. It is easy to verify that the space *X* so defined is Boolean. For instance, a subset of *X* is clopen if and only if it is either a finite subset (of *X*) that does not contain $\{x_0\}$ or else a cofinite subset that contains x_0; indeed, a subset and its complement are both open just in case one of them (the one that contains x_0) is cofinite. The clopen sets form a base for the topology because every open set that contains x_0 is clopen, while every open set that does not cotnain x_0 is the union of its finite subsets." (Givant and Halmos [2009] p. 301, where the discussion continues.) We also have Alexandroff's Theorem: Let *X* be a locally compact space. (1) Then there exists a compact space X_∞ and a homeomorphism *f* of *X* onto the complement of a point $x_0 \in X_\infty$. (2) If X_∞' is another compact space such that there is a homeomorphism f_1 of *X* onto the complement of a point in X_∞', then there is a unique homeomorphism *g* of X_∞ onto X_∞' such that $f_1 = g \circ f$. [Source is Bourbaki [1989a] p. 92]

Before proceeding further, we should examine precisely why this definition results in a compactification of $\mathbb{N}$. This derives from the definition of the topology on $\mathbb{N}_\infty$.



The topology is defined in such a way that, if an open set, U_∞, covers $\{\infty\}$ then what remains, $\mathbb{N} - U_\infty$, must be finite; hence every open cover has a finite subcover.

If we now pair-off elements of $\mathbb{N}_\infty$ with elements of ω we obtain a one-one correspondence with ω in correspondence with the final partition, $\{\infty\}$. However, we should be careful here as well. Although $\mathbb{N}_\infty$ and ω are equinumerous it would be a mistake to conclude that they *represent the same structure* in regard to the partition of the interval $[0,1]$ into actually ω parts. The 1-point compactification $\mathbb{N}_\infty$ generates the following image: -



But we must take another view of this partition of $[0,1]$ if we wish to use it as a basis of a formal analytic logic. The reason is that a formal analytic logic constructs a lattice over a partition (its skeleton) but in such a way that there is no regard as to the order relations of the partitions. The dilution inference $\alpha_i \vdash \alpha_i \vee \alpha_j$ where α_i, α_j are atomic propositions corresponding to atoms $\{i\}, \{j\}, i \neq j$ makes no reference to order whatsoever: the inference does not depend on α_i coming before α_j. (In fact, we *insist* upon it when we make $\alpha_i \vee \alpha_j \equiv \alpha_j \vee \alpha_i$.) Hence, we need yet a third notion of the infinite, to denote a potentially infinite collection of unordered sets – a potentially infinite antichain [See 4 / 11.1]. I shall denote this collection by μ; μ is $\mathbb{N}$ without any order relation upon it. It is what you would have if you did not have complete induction defined on $\mathbb{N}$, which is a potentially infinite chain.

3.10 (+) Summary

ω Actually infinite chain of ordinals.

$\mathbb{N}$ Potentially infinite chain of natural numbers.

μ Potentially infinite antichain of unordered elements.

The status of μ and $\mathbb{N}$ as sets is disputable. According to the popular theory that every mathematical entity is a set, they must either be sets or just not exist. Levy [2002] does not include $\mathbb{N}$ in his text at all, which suggests that he denies that it exists as a separate entity from ω. However, we can see that the one-point compactification of $\mathbb{N}$ makes no sense at all if $\mathbb{N}$ does not exist and the existence of $\mathbb{N}$ *as distinct from* $\mathbb{N}_\infty$ is implicit in the entire theory of Boolean lattices. So there is the question: should we identify ω with $\mathbb{N}_\infty$? The answer is that strictly speaking, as intensions, we have as distinct concepts: -

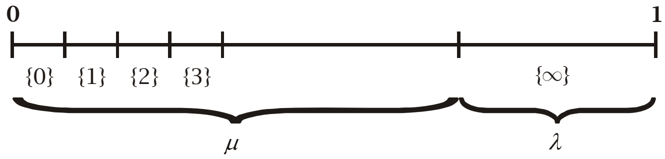
3.11 (+) Distinct variants of the partition of the interval

1. $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\} = \{0,1,2, \ldots , \infty\}$ is the one-point compactification of $\mathbb{N}$ and a model of the actually infinite partition of the interval $[0,1]$.
2. ω is the least limit ordinal – an actually infinite collection of all finite ordinals: $\omega = \{0,1,2,3, \ldots\}$. There is no ∞ in this set.[16]
3. $\mu_\infty = \mu \cup \{\infty\} = \{0,1,2, \ldots , \infty\}$ is the partition of the interval $[0,1]$ in which μ represents an antichain. Over this partition we construct a Boolean lattice in which every singleton set is 1 unit in the metric from the **0** of the lattice.

All of these partitions are countably infinite and equinumerous to ω. Here I shall adopt the view that it is always a mistake to identify $\omega = \mathbb{N}$; however, for reasons that shall become apparent below [See 3.18 and following] it shall emerge that μ and $\mathbb{N}$ are different descriptions of the same underlying structure, so we shall allow $\mu = \mathbb{N}$, subject to caveats. By the same token, the Boolean algebra that we are interested in is strictly $\mathbf{2}^{\mu_\infty}$, but it is isomorphic to the Cantor set, hence is the Cantor space, and we shall allow, $\mathbf{2}^{\mu_\infty} = \mathbf{2}^{\mathbb{N}_\infty} = \mathbf{2}^{\omega}$, also subject to caveats. We shall use the canonical $\mathbf{2}^{\omega}$ for the Cantor set, which is the main structure under scrutiny in this inquiry, at least, so far as the mathematics is concerned.

The partition of $[0,1]$ into a potentially infinite antichain followed by a point at infinity, $\{\infty\}$, may be pictured thus: -

[16] The distinctions between $\mathbb{N}, \mathbb{N}_\infty$ and ω raise the question of non-standard models of arithmetic in which we see additional elements tagged onto the set $\mathbb{N}$ and making the resultant model non-categorical for $\mathbb{N}$. This relates to ω-consistency. For a description of non-standard models of arithmetic, see Boolos and Jeffrey [1980] Chapter 17.



This now partitions the line into atoms, which are labelled $\{0\},\{1\},\{2\},\ldots$ followed by $\{\infty\}$. The labels are for the purpose of identifying different segments of the interval – separating them. It is essential that the collection, $\{0\},\{1\},\{2\},\ldots$, be treated as an unordered one, since if it is ordered then it becomes a chain and not an antichain and the lattice generated by it is also a chain and not a Boolean lattice. I have also labelled $\lambda=\{\infty\}$. We observe that since $\mu\cup\lambda=[0,1]$ we have $\lambda=\mu'$ - they are mutual complements in the interval. Hence, $\{\mu,\lambda\}\cong\{0,1\}=\mathbf{2}$. Observe also: $\mu\cup\lambda=\{0,1,2,3,\ldots\}\cup\{\infty\}=\{0,1,2,3,\ldots,\infty\}$ as an unordered antichain. μ is intrinsically unordered but it may be well-ordered extrinsically by placing it into one-one correspondence with elements of $\mathbb{N}$. However, this assumes the well-ordering principle, which is equivalent to the Axiom of Choice: -

3.12 Axiom of choice

To any nonempty set S who elements are nonempty sets S_α there exists a function $f:S\to\bigcup_{S_\alpha\in S}S_\alpha$ such that $f(S_\alpha)\in S_\alpha$ for all $S_\alpha\in S$. The function is called the *choice function.*

3.13 Definition, well-ordered

Let $(X,\leq)$ be a totally ordered set. Then X is said to be well ordered if and only if every non-empty subset Y of X contains a minimal element; that is, there exists an element $y\in Y$ such that for all $x\in X$, $y\leq x$. This element y is said to be the least element of *Y.*

3.14 Well-ordering principle

Every set can be well-ordered.

3.15 Zorn's lemma

Let X be a partially ordered set in which every chain (i.e totally ordered subset) has an upper bound, then X possesses a maximal element.

3.116 Hausdorff maximality principle

Let $(X,\leq)$ be a partially ordered set, and let T be the set of all totally ordered subsets of *X.* Suppose that T is partially ordered by inclusion, $\subset$. Then $(T,\subset)$ has a maximal element.

3.17 Result

The following four axioms are equivalent.

1. Axiom of choice
2. Zorn's lemma
3. Hausdorff maximality principle
4. Well-ordering principle

This result, which is standard in set theory, is constructed on the implicit assumption that $\omega = \mathbb{N}$ and that there is no distinction between the potential and actual infinite. If, indeed, $\omega \neq \mathbb{N}$, these equivalences will have to be revisited. I conjecture that there are two collections of equivalences: one for potential infinities that do not have maximal elements, and another for actual infinities that do.

Another equivalence to the Axiom of Choice is: -

3.18 Rubin's proposition

(Levy [2002] p. 164 – original result by H. Rubin 1960) The axiom of choice is equivalent to the statement "The power set of every ordinal is well-orderable".

Application of the Axiom of Choice transforms μ into a well-ordered set and transforms the partition $\mu \cup \lambda$ into a set isomorphic to ω.

$$\mu \xrightarrow[\text{Transformation}]{\text{Axiom of Choice}} \mathbb{N} \qquad \mu \cup \lambda \xrightarrow[\text{Transformation}]{\text{Axiom of Choice}} \omega \cong \mathbb{N} \cup \{\infty\}$$

3.19 The relationship of the Axiom of Choice to the Axiom of Completeness

In technical textbooks of set theory I see no formal treatment of the Axiom of Completeness. For example, Levy [2002] does not mention it. However, the Completeness Axiom is implicit everywhere and an equivalent to it is needed whenever a recursive finite process needs to be completed, just on analogy with the Dedekind cut. Whenever this is required it is always the Axiom of Choice that is invoked within the context of ZF theory as a whole, which already has the Axiom of Infinity. But the Axiom of Infinity is not enough to supply completeness arguments. Hence, I conjecture that the Completeness Axiom is expressed in ZFC by the combination of the Axiom of Infinity with the Axiom of Choice. To support this view consider the following remark from Givant and Halmos [2009]: -

> There is a close connection between complete ideals and the "cuts" that play a crucial role in Dedekind's classical construction of the real numbers from the rational numbers. (Givant and Halmos [2009] p.206)

The complete ideals that they discuss can only be established on the basis of the Axiom of Choice. They are the subject of the mathematical parts of this paper.

3.20 The relationship of the Axiom of Choice to complete induction

Furthermore, the Axiom of Infinity provides a principle of induction on chains in set theory – which goes by the apparently "strong" title of "transfinite induction". But since not all sets are known to be chains, the Axiom of Infinity must be supplemented by the statement that *all sets can be well-ordered*, in other words, that *all sets are chains*, which is supplied by the Axiom of Choice. Hence, we see that the Axiom of Choice is equivalent to complete induction on actually infinite sets and is the analogue in set theory to the principle of complete induction on potentially infinite sets in arithmetic. We see that in set theory the Axiom of Completeness follows from the presence of an *Axiom of Complete Induction on actually infinite sets*, which I propose should be the *correct* description of the Axiom of Choice. To support this conclusion consider the following quotation from Levy [2002]: -

> **Zorn's Lemma eliminates Recursion.** In applications Zorn's lemma often replaces the sue of definition by recursion of a function H where one proceeds as long as some condition is met ... in situations where at the recursion step one may have to chose $H(\alpha)$ as an arbitrary member of some set.

Definition by recursion is an expression of the inductive argument. So here we see the Axiom of Choice taking the place and doing the work of complete induction within set theory.

3.22 (+) Result

The definition of the topology T on $\mathbb{N}_\infty$ [3.5 above] makes an implicit use of the Axiom of Choice [3.7 above].

<u>Proof</u>

The construction: -

$$\infty \in U \text{ and } \mathbb{N}_\infty - U \text{ is finite} \quad \Rightarrow \quad U \text{ is open in } T$$

requires that we are able to pick out the element ∞ from $\mathbb{N}_\infty = \{0,1,2,3, \dots, \infty\}$ which is not possible unless we have a choice function, or equivalently, unless $\mathbb{N}_\infty$ is an well-ordered set. The collection $\mathbb{N}_\infty = \mu \cup \lambda$ is primarily an unordered anti-chain. It requires the Axiom of Choice to give it an alternative description as an ordered chain.

We shall see below [Chapter 6, Section 1 et seq.] that the topology $(\mathbb{N}_\infty, T)$ induces a compact topology on the Cantor set. That the Cantor set is compact is equivalent to Tychanoff's theorem [6.1.3], which is itself derived from the Axiom of Choice. Hence, compactness of $\mathbb{N}_\infty$

must assume a principle at least equivalent in deductive strength to Tychanoff's theorem, and the above proof demonstrates this.

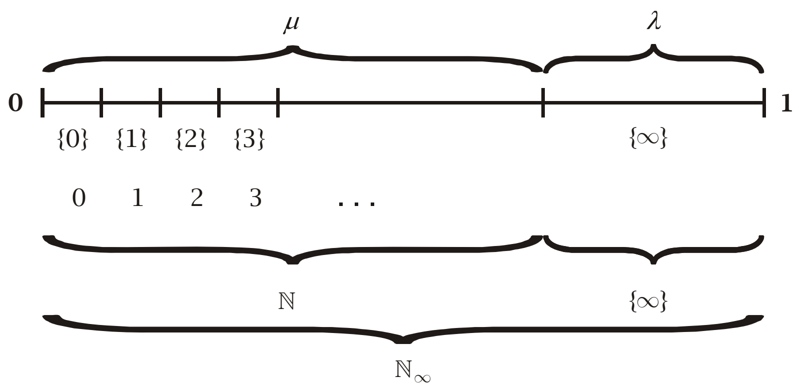
3.23 The analytic logic of the actually infinite partition

The construction of the analytic logic of actually ω parts takes place in two "conceptually" distinct stages. Firstly, take an unordered partition: -

$\mu \cup \lambda = \{0,1,2,3, ...\} \cup \{\infty\} = \{0,1,2,3, ..., \infty\}$

and from this generate the lattice $\mathbf{2}^{\omega}$; then "after the fact" we apply the Axiom of Choice to determine that there is an exogenous order relation on the skeleton [Chap. 4, Sec. 11.3], that also enables us to see the element λ as last in the succession of all the other elements contained in μ which is now identified with $\mathbb{N}$. Implicit in all the process is the assumption that μ and $\mathbb{N}$ are sets. If they are not sets then they cannot be well-ordered. We also conclude that μ and $\mathbb{N}$ are the same set but under different descriptions. This is what the Axiom of Choice does in general – it changes the description of a set. Another way of interpreting this is that $\mathbb{N}$ is an ordered set (under successor) but in order to construct the lattice over it, we *forget* its chain structure, and so arrive at an unstructured antichain, $\mathbb{N}_\infty = \mu \cup \lambda$. Once we have built the lattice over this, we then *remember* its chain structure and impose that in some way on the lattice, after the fact.

This is not exactly "set theory" in the "classical" sense, so I must stress that all I am doing here is uncovering the principles behind one-point compactification, which has already been introduced into the literature and which clearly distinguishes between $\mathbb{N}$ on the one hand and $\mathbb{N}_\infty$ on the other. The other point is, for the purposes of formal analytic logic, to keep a clear separation between the antichain properties of the Boolean lattice and the chain properties of $\mathbb{N}$.



If μ is a set, then the power set operation may be defined upon it. Suppose the elements of μ are numbers whose internal properties are remembered (for example, their prime factorisations) but whose external succession in a chain is temporarily forgotten. Then $\mathbf{P}(\mu)$ is the set of all finite subsets of natural numbers, $n \in \mathbb{N}$. Denote this set by $\mathrm{M} = \mathbf{P}(\mu)$; this shall be shown to represent the maximal ideal of all finite subsets of $\mathbb{N}$. [Chapter 6, section 2 et seq.]

As this introduction of the power set operation represents another conceptual step forward, I should dwell upon it further. The power set operation is defined on ω, and we see $\mathbb{R} = \mathbf{P}(\mathbb{Q}) \cong \mathbf{P}(\mathbb{N}) \cong \mathbf{2}^{\omega}$ where $\mathbf{2}^{\omega}$ denotes cardinal exponentiation. (This is a major theorem of Cantor allowing the usual identification of real numbers with their decimal expansions.) Under the (false) identification of $\omega = \mathbb{N}$ we obtain $\mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$. Now if we allow $\omega \neq \mathbb{N}$, and also allow $\mathbb{N}$ to be a set, then we have $\mathbf{P}(\mathbb{N}) \neq \mathbf{P}(\omega)$. We have $\mathbf{P}(\mathbb{N})$ as a <u>proper subset</u> of $\mathbf{P}(\mathbb{N}_{\infty}) \cong \mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$. Since $\mathbb{N}$ is a potentially infinite unbounded collection we cannot obtain a collection of real numbers by taking the power set upon it; the real numbers are generated by the actually infinite partition of the interval, $\mathbb{N}_{\infty}$. Hence we have: -

3.24 (+) The power set operation on potentially and actually infinite collections

1. $\mathbb{R} = \mathbf{P}(\mathbb{N}_{\infty}) \cong \mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$
2. $\mathrm{M} = \mathbf{P}(\mathbb{N}) \cong \mathbf{P}(\mu)$

4 Atoms

Intuitively, in any lattice an atom is an element of the lattice that is 1 unit of measure away from the **0** element. An atom is said to "cover" the **0**. [See 4.5.3] Halmos [1963] defines an *atom* of a Boolean algebra to be an element that has no non-trivial proper subelements. All finite lattices must be atomic, because of the discrete nature of the partition of the underlying space.

4.2 Definition, atomic

A Boolean algebra is said to be *atomic*, iff, for every non-zero element *x* of the algebra, there is some atom α such that $\alpha \leq x$.

4.3 Definition, non-atomic

A Boolean algebra is said to be *non-atomic* if it has no atoms.

Lattices are built over partials orders.

4.4 Definition, dominates

If in a partial order we have the relation $x \leq y$ then we say that *x* lies *below y*, and *y dominates x*.

Using this term *dominates*, we may give an equivalent definition of an atomic lattice.

4.5 Definition, atomic algebra

A Boolean algebra is said to be *atomic* if every non-zero element dominates at least one atom.

It is remarked by Givant and Halmos [2009] "that these two concepts are not just the negations of one another." To say that an algebra is "non-atomic" is not the same as saying that it has no atoms. Broadly speaking a Boolean algebra can fail to be atomic in two ways: -

1. It can have no atoms at all. This occurs in the case of the interval algebra. [6 / 2.7 and 5.12 below]
2. It can have *notional* atoms [See section 5.8 below, and following] but not all of them.

The term *non-atomic* refers to the first case, and the term *atomic* means that the algebra has enough atoms so that every element dominates at least one atom.

4.6 Lemma

The following conditions on an element q in a Boolean algebra are equivalent: -

1. q is an atom.
2. For every element p, either $q \le p$ or $q \wedge p = 0$, but not both.
3. For every element p, either $q \le p$ or $q \le p'$, but not both.
4. $q \ne 0$, and if q is below a join $p \vee r$, then $q \le p$ or $q \le r$.
5. $q \ne 0$, and if q is below the supremum of a family $\{p_i\}$, then q is below p_i for some i.

(See Givant and Halmos [2009] for proofs.)

4.7 Criterion, non-atomic

A Boolean algebra B is non-atomic if for all $b \in B$ there exist an element $a \ne 0, a \in B$ such that $a \cdot b = a$; alternatively, such that $a \cdot b = 0$. (Komjáth and Totik [2000])

4.8 Lemma

In an atomic algebra every element is the supremum of the atoms it dominates. (See Givant and Halmos [2009] for proof.)

4.9 Lemma

If an element p in a Boolean algebra is the supremum of a set of atoms E, then E is the set of all atoms below p. (Komjáth and Totik [2000] p. 11, qn. 34)

Let us recall the definition of a complete lattice.

4.10 Definition, join, meet of a subset

Let A be a subset of a Boolean algebra B. If the supremum of A exists it is denoted $\bigvee_{x \in A} x$ and is called the *join* of A. If the infimum of A exists it is denoted $\bigwedge_{x \in A} x$ and is called the *meet* of A.

4.11 Definition, complete Boolean algebra

A Boolean algebra B is said to be *complete* if and only if $\bigvee_{x \in A} x$ and $\bigwedge_{x \in A} x$ exist for all subsets A of B.

Any atomic lattice must also be complete, and we have already proven this for the finite case.

4.12 Definition, ring of sets, field of sets

A *ring* of sets is a family Φ of subsets of a set I which contains with any two sets S and T also their (set-theoretic) intersection $S \cap T$ and union $S \cup T$. A *field* of sets is a ring of sets which contains with any S also its set-complement S'. (Birkhoff [1940])

4.13 Definition, complete field of sets

A *complete field of sets* is a field of sets such that, for any subset A of the field, the union and intersection of the sets in A are also in the field.

It is essential to distinguish a field of sets from the power set of a set X. Certainly, if X is a non-empty set then $\mathbf{P}(X)$ is an atomic, complete field of sets, *but the converse does not apply.* In other words, there can be a complete field of sets that is *not a power set.*

4.14 Example

In $\mathbf{2}^4$ the collection, $\{\varnothing,\{1,2\},\{3,4\},\{1,2,3,4\}\}$ is a complete field of sets that is not the power set of some subset of $X = \{1,2,3,4\}$.

This field is also atomic, with atoms $\alpha_1 \equiv \{1,2\}$ and $\alpha_2 \equiv \{3,4\}$. This illustrates the result that any complete field of sets is atomic.

4.15 Result

A complete field of sets is a complete Boolean algebra wherein

$\bigvee_{x \in A} x = \bigcup_{x \in A} x$ and $\bigwedge_{x \in A} x = \bigcap_{x \in A} x$.

4.16 Example

A singleton in a field of sets is an atom.

4.17 Result

Any complete field of sets F is atomic

Proof

Let A be any non-empty set belonging to F and let $x_0 \in A$. Let $H = \bigcap$subsets of F containing x_0. By completeness $H \in F$. To show that H is an atom: let $W \subseteq H$ and $W \in F$. Then either (1) $x_0 \in W$ implies $H \subseteq W$, so $H = W$; or (2) $x_0 \notin W$ implies $x_0 \in H - W \in F$. Then $H \subseteq H - W$ implies $W = \varnothing$.

4.18 Result

Let A be the set of atoms of a Boolean algebra B. Then B is atomic iff $\bigvee_{x \in A} x = 1$.

Proof

I. Forward derivation. If B is atomic then $\bigvee_{x \in A} x = 1$.

II. Reverse derivation. Let $\bigvee_{x \in A} x = 1$. Let $y \neq 0 \in B$. We have to show that there exists an atom $b \leq y, b \in B$. Suppose not. Then let x be an atom. Then $x \wedge y = 0$. This mplies $\bigvee_{x \in A}(x \wedge y) = \mathbf{0}$. But $\bigvee_{x \in A}(x \wedge y) = y \wedge \bigwedge_{x \in A} x = y \wedge 1 = y$. Hence, $y = 0$ contradicting $y \neq 0$.

5 The atomless Boolean algebra of statement bundles

The Boolean algebra of statement bundles was defined in Chapter 4 section 7, where it was called the Tarski-Lindenbaum algebra. Here we now show that it is atomless.

5.1 Result

Every field of subsets $\mathbf{P}(X)$ of some non-empty set X is complete. [defined 4.11]

Proof

$\bigvee_{x \in A} x = \bigcup_{x \in A} x$ and $\bigwedge_{x \in A} x = \bigcap_{x \in A} x$.

By 4.15 above $\mathbf{P}(X)$ is complete.

5.2 Corollary

The theorem: Let X be a finite or infinite set. Then the Boolean algebra $\mathbf{P}(X)$ is atomic.

This means that the Cantor set, $\mathbf{2}^{\omega} = \mathbf{P}(\omega)$ is complete and atomic. The collection of all finite subsets of $\mathbb{N}$, which shall be denoted $F(\mathbb{N})$, is a countably infinite collection. Thus $F(\mathbb{N})$ is a proper subset of the Cantor set, $\mathbf{2}^{\omega} = \mathbf{P}(\omega)$.

5.3 Result

Any countably infinite Boolean algebra is not atomic.

Proof

Let *B* be a countably infinite Boolean algebra, that is of cardinality $\aleph_0$, and suppose *B* is atomic. Let *A* be the set of all atoms of *B* and by the isomorphism theorem $B = \mathbf{P}(A)$. *A* cannot be finite, for then *B* would be finite. Therefore, *A* is countably infinite. Then $P(A)$ is an atomic Boolean algebra of cardinality $2^{\omega} > \omega$. This contradicts the assumption that *B* is of cardinality ω. Hence, *B* cannot be atomic.

5.4 Result

The Boolean algebra, *S*, of (propositional) statement bundles is not isomorphic to some $\mathbf{P}(A)$.

Proof

Since the set of statements is denumerable, and the statement bundles are equivalence classes defined upon them, then the set of statement bundles is also denumerable. Suppose $S \cong \mathbf{P}(A)$ for some set *A*. But then *A* must be infinite, for so is *S*. The least cardinality *A* can have is $\aleph_0$, whence the cardinality of $|\mathbf{P}(A)| = 2^{\aleph_0} > \aleph_0$ by Cantor's theorem. [Chap.2 / 2.7.6].

5.5 Corollary

The Boolean algebra *S* of (propositional) statement bundles is atomless.

Proof

The Boolean algebra *S* is countably infinite. Hence by the preceding theorem it is atomless.

5.6 Result

Every atomless Boolean algebra with more than one element must be infinite.

Proof

The unit 1 is different from zero, so there is a non-zero element p_1 strictly below 1; otherwise, 1 would be an atom. Because p_1 is not zero, there must be a non-zero element strictly below p_1; otherwise, p_1

would be an atom. Continue in this fashion to produce an infinite, strictly decreasing sequence of elements $1 > p_1 > p_2 > \ldots$.

5.7 Result

Any two countably infinite Boolean algebras (without atoms) are isomorphic.

Proof

It can be shown that the order of any finite Boolean algebras is 2^n for some $n \in \mathbb{N}$, and any two finite Boolean algebras with the same number of elements are isomorphic.

Let *A, B* be two countably infinite Boolean algebras. Let $A = \{a_0, a_1, a_2, \ldots\}$ and $B = \{a_0, a_1, a_2, \ldots\}$ be enumerations of *A* and *B* respectively. The proof will be a "back and forth" argument.

The proof proceeds inductively on the order of subalgebras.

For $n = 0$ let $A_0 = \{0,1\}$ and $B_0 = \{0^*, 1^*\}$; then $A_0 \cong B_0$.

For the induction step, suppose that the result is true for all $k < n$; that is $A_k \cong B_k$.

Suppose *n* is even.

Let $a_j \in A - A_{n-1}$ be the element with smallest index *j*, and let A_n be the subalgebra generated by $\{a_j\} \cup A_{n-1}$.

Then there is an element $b_m \in B - B_{n-1}$ such that the subalgebra B_n generated by $\{b_m\} \cup B_{n-1}$ is isomorphic to A_n.

Suppose *n* is odd.

In this case select first $b_m \in B - B_{n-1}$ to generate B_n, and the claim is that there is an element $a_j \in A - A_{n-1}$ that generates $A_n \cong B_n$.

So the induction step holds and so $A = \bigcup A_n \cong \bigcup B_n = B$.

We would also have to prove that both algebras pair off atoms in the respective finite subalgebras. (This proof is based on Komjáth and Totik [2000]. There is also a proof in Givant and Halmost [2002] p.135)

5.8 The algebra of all finite subsets of natural numbers

Let us now interpret these results in the light of what we have learnt about the 1-point compactification of $\mathbb{N}$, which we treated as the compactification of the antichain μ, having the same elements as $\mathbb{N}$ but having "forgotten" its chain structure. The compactification creates a closed, bounded set, $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ that is equinumerous to ω; hence the lattice generated by this skeleton is the Cantor set, $\mathbf{2}^{\omega}$. We see that $F(\mathbb{N})$, which is the ideal in $\mathbf{2}^{\omega}$ of all finite subsets of $\mathbb{N}$ is a countably infinite algebra in its own right. About this structure, Birkhoff writes: -

> Stone has defined a *generalized Boolean algebra* as a *relatively* complemented distributive lattice with an *O* (but not necessarily an *I*) such as the following: The lattice $\mathbf{2}^{(\omega)}$ of all *finite* subsets of the set $\mathbb{Z}^{+}=\omega$ of positive integers is a generalized Boolean algebra. The corresponding characteristic functions (with values in $\mathbb{Z}_2$) form a Boolean ring, which is the restricted direct sum (product) of countably many copies of $\mathbb{Z}_2$. Note that $\mathbf{2}^{(\omega)}$ is an ideal in $\mathbf{2}^{\omega}$, the set of *all* subsets of ω; the quotient-lattice $\dfrac{\mathbf{2}^{\omega}}{2^{(\omega)}}$ has many interesting properties. (Birkhoff [1940] p.49)

Stone introduced the symbol $\mathbf{2}^{(\omega)}$ for $F(\mathbb{N})$. $\mathbf{2}^{<\omega}$ is also seen in the literature, and this shall be used here in preference because of the unbounded nature of $F(\mathbb{N})$. I shall write $F(\mathbb{N})\cong\mathbf{2}^{<\omega}$, which is also standard practice, because I regard $\mathbf{2}^{<\omega}$ as the underlying structure to which many other structures are isomorphic. For example, the set of all cofinite subsets of $\mathbb{N}_{\infty}$, denoted $C(\mathbb{N}_{\infty})$ is isomorphic to $\mathbf{2}^{<\omega}$. Any countable number of products or sums of $\mathbf{2}^{<\omega}$ is isomorphic to $\mathbf{2}^{<\omega}$. The Boolean algebra of all finite and cofinite subsets of $\mathbb{N}_{\infty}$, denoted $FC(\mathbb{N}_{\infty})\cong FC(\omega)$ is also isomorphic to it.[17]

The above results show that $F(\mathbb{N})\cong\mathbf{2}^{<\omega}$ is atomless, and this would appear puzzling. The reason why $F(\mathbb{N})$ is atomless is because $\mathbb{N}$ is potentially, not actually, infinite. It would seem that in the singleton sets $\{0\}\leftrightarrow 0\in\mathbb{N}, \{1\}\leftrightarrow 1\in\mathbb{N}, \ldots$ we have ideal candidates for atoms of the resultant algebra, all the more so because the corresponding sets based on ordinals $\{0\}\leftrightarrow 0\in\omega, \ldots$ are atoms of the atomic, complete Boolean algebra $\mathbf{2}^{<\omega}$ of which $F(\mathbb{N})$ is a proper subset. However, the problem is resolved when we realise that $F(\mathbb{N})$ may be regarded as being always in a state of ongoing generation that has never been completed. Suppose we take it as completed as some finite stage of its generation; then what we obtain is a finite set of *notional atoms*: $A=\{\{0\},\{1\},\ldots,\{n\}\}$ and the finite Boolean algebra $\mathbf{2}^{n}$. It is important to realise that this is *atomic in itself* but as a representation of $F(\mathbb{N})$ always incomplete. No finite determination of *notional* atoms of $F(\mathbb{N})$ is ever sufficient to capture its potentially

---

[17] $FC(\mathbb{N}_{\infty})$ is an algebra in this case and not merely a generalised one, because it contains the topmost element 1 of the algebra $\mathbf{2}^{<\omega}$.

infinite structure. A determination of notional atoms, $A = \{\{0\},\{1\}, \dots ,\{n\}\}$, may potentially be enlarged in one of two equivalent ways: either (1) we had further atoms to the end of the list, $\{n+1\}, \{n+2\}, \dots$, or we allow meets of the notional atoms to be defined: $\{0\} \wedge \{1\}, \{0\} \wedge \{2\}, \dots$; the formation of these shows that the notional atoms never were true atoms in the first place; the two approaches are equivalent because the second of these would result in a re-labelling of the notional atoms to new notional atoms: $\{0\}^* = \{0\} \wedge \{1\}$, $\{1\}^* = \{0\} \wedge \{2\}, \dots$; $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ is an algebra that possesses *notional* atoms rather than real ones. They are *notional* because any finite determination of $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ is such that it may be embedded in another larger lattice with new notional atoms that subsume the previous ones. This process of replacing one determination of $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ by a larger one, in which it is embedded, I shall call *lowering the floor of the lattice*. [Chap.5 /5.8 and Chap. 7 /1.6] This is on the principle that the notional atoms are being shown to not really be atoms by the formation of new meets lying "below" them in the extended lattice. Hence $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ is not a determinate Boolean algebra but rather a potentially infinite collection of finite algebras: $F(\mathbb{N}) = \mathbf{2} \subset \mathbf{2}^2 \subset \mathbf{2}^4 \subset, \dots , \subset \mathbf{2}^{2^n} \subset \mathbf{2}^{2^{n+1}} \subset, \dots$. What was denoted by $\mathrm{M} = \mathbf{P}(\mu)$ above [Sec. 3.24] is the supremum of this sequence. Because $\mathbb{N}$ is locally compact, so is $F(\mathbb{N})$; likewise, we see that each member of the sequence above is compact (*a fortiori* locally compact), but $F(\mathbb{N})$ itself is not compact because it is not bounded above. The *maximal ideal* is the completion of $F(\mathbb{N})$.

5.9 (+) The ideal and maximal ideal of all finite subsets of natural numbers

1. $F(\mathbb{N}) = \mathbf{P}(0) \cup \mathbf{P}(1) \cup \mathbf{P}(2) \cup \dots$

   Potentially infinite ideal of all finite subsets of natural numbers.

2. $\mathrm{M} = \max\{\mathbf{P}(0) \cup \mathbf{P}(1) \cup \mathbf{P}(2) \cup \dots\} = \mathbf{P}(\mu) \cong \mathbf{P}(\mathbb{N})$

   Actually infinite maximal ideal of all finite subsets of natural numbers.

The relationship of $F(\mathbb{N})$ to M is as $\mathbb{N}$ to ω.

5.10 (+) Definition, interior of maximal ideal

I shall call $F(\mathbb{N})$ the *interior* of M.

For $\mathbb{N}_\infty = \{0,1,2, \dots \infty\}$ we can form singleton sets, $\{0\},\{1\},\{2\}, \dots ,\{\infty\}$ and identify these with the atoms. So the Boolean algebra $\mathbf{2}^{\mathbb{N}_\infty} \cong \mathbf{2}^{\omega}$ is atomic. Being atomic has an orthogonality property: if α_1, α_2 are any two atoms then $\alpha_1 \wedge \alpha_2 = 0$; this confirms that any singleton set can be identified with an atom. Atoms may also be represented by prime numbers in the lattice of divisors where: -

$\vee$ = least common multiple and $\wedge$ = greatest common divisor

For any two primes $p_1 \wedge p_2 = 0$, so atoms may be regarded as prime factors of the lattice. The set $\mathbb{N}_\infty = \{0,1,2, \dots \infty\}$ acts as a reference set for atoms for an atomic Boolean lattice $\mathbf{2}^\omega$ because every element within it can be put into one-one correspondence with a prime, *excepting* the "point at infinity" ∞. In a countable non-atomic Boolean lattice there are no true singleton subsets, and any subset appearing as a singleton subset is merely a notional atom or notional prime.

5.11 Countable collection of prime generated infinite and coinfinite sets

Let $PI(\mathbb{N}_\infty) \cong PI(\omega)$ denote the countable collection of prime generated infinite and coinfinite sets – sets formed from $\omega \cong \mathbb{N}_\infty$ by deleting sets of infinite size that are equivalence classes of numbers divisible by a given prime to obtain other sets of infinite size. Specifically, denote the set of all numbers divisible by prime p by $[p]$. For example: -

$[2] = \{0,2,4,6,\dots\}$ is the actually infinite set of all even numbers.

Then we obtain ideals in $PI(\omega)$ by the difference of ω and any equivalence class $[p]$. For example: -

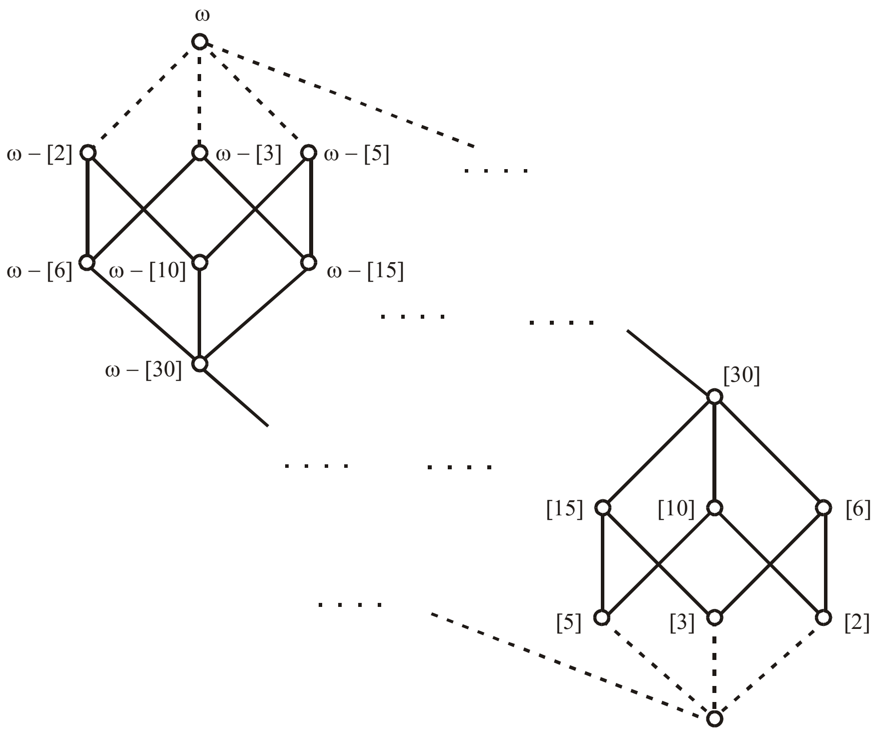
$[1] = \omega - [2] = \{1,3,5,7,\dots\}$ is the set of all odd numbers.

Since $[2]$ is a member of the lattice $\mathbf{2}^\omega$, then its complement $[1] = \omega - [2]$ is a member of the lattice. This lattice $PI(\omega)$ is isomorphic to $FC(\mathbb{N}_\infty) \cong FC(\omega)$, which contains the singleton sets $\{k\}$ for all $k \in \omega$, and the Boolean primes [See 7.19 below] $\omega - \{k\}$. Likewise, $\omega - [2]$ is **not** a prime and $[2]$ is not an atom of $2^{<\omega}$, which is atomless and likewise, lacks its Boolean primes. The structure of $PI(\omega)$ is simultaneously generated downwards from ω and upwards from Boolean zero. It meets in the middle because any element in the middle can be reached after a countable number of joins, meets and relative complements.

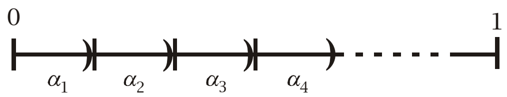
These sets are equinumerous to $\omega = \{1,2,3,\dots\}$; they are infinite sets and members of the algebra of infinite and co-infinite subsets of ω. The complements are always also infinite sets as well. By taking intersections and complements we obtain "smaller" sets – in the sense of proper subsets, but these are also infinite subsets of infinite sets. The intersection of any two of these sets is an infinite set. We never break out of infinity and to do so would require ω operations, equivalent to taking a limit. Examples of these operations are: -

$$x_0 = [2] = \{0,2,4,6, \dots\} \qquad y_0 = [2] - [6] = \{2,4,8,10, \dots\}$$
$$x_1 = [6] = [2] \cap [3] = \{0,6,12, \dots\} \qquad y_1 = [30] - [6] = \{6,12,18,24, \dots\}$$
$$x_2 = [30] = [6] \cap [5] = \{0,30,60, \dots\} \qquad y_0 \wedge y_1 = y_0 \cap y_1 = \varnothing$$

We are “stuck in the middle” of the lattice – in what may be called the *boundary region* of $\mathbf{2}^{\omega}$. We have $PI(\omega) \cong \mathbf{2}^{<\omega}$ which is countably infinite and atomless.



5.12 Model of the countable collection of intervals identified by their end-points
(Givant and Halmos [2009] p.25 et seq.)
Left half-closed intervals: $[a,b) = \{x \in X : a \le x < b\}$ define an *interval algebra*.



It is a countable collection of intervals identified by one end points. Each interval is an open subset of $[0,1]$ and hence atomless.

The *degenerate algebra* is the one-element Boolean algebra. This is “vacuously atomless” because it has no non-zero elements. About the interval algebra, Givant and Halmos comment: -

> Interval algebras provide examples of non-degenerate atomless Boolean algebras. For instance, the interval algebra of the real numbers is atomless, and so is its subalgebra consisting o f the finite unions of left half-closed intervals

with endpoints that are rational numbers (or $\pm\infty$). Notice that this last algebra is countable. Quite surprisingly, it is the only possible example of a countable atomless Boolean algebra that is not degenerate, at least up to isomorphic copies (Givant and Halmos [2009] p.134.)

The intervals are connected and hence do not provide atoms. If either we strive to create totally disconnected sets by taking countable intersections of these intervals and so reach down to prime ideals = atoms, or we strive to create larger intervals by taking unions of these intervals and so built up to maximal filter, neither operation can be completed. We are always left with sets that contain infinite collections of points in both directions. This is a model of the atomless algebra.

5.13 Result, the quotient algebra $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ is atomless.

Proof

Let $[x] \neq [0]$ denotes any equivalence class in $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$; that is $[x] \in \frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$. Then $x \in \mathbf{2}^{\omega}$ is an infinite set. Since x is an infinite set it contains a countable number of infinite subsets that can be arranged in a chain. That is, there is also an infinite set $y \subseteq x$ such that $x - y$ is infinite. Since y is infinite $y \neq 0_{\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{(\omega)}}}$. Since $y \subseteq x$ we have $[y] \leq [x]$. Since y is infinite $x + y \notin \mathbf{2}^{<\omega}$; hence $x \not\equiv_{\mathbf{2}^{(\omega)}} y$ and $[x] \neq [y]$. So we have $0_{\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{(\omega)}}} < [y] < [x]$. That is, $[x]$ is not an atom. By generalisation, no equivalence class $[x] \neq 0 \in \frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ can be an atom. In $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ the equivalence classes are formed by taking any infinite set x and adding to that set any finite set (that is, any set $r \in \mathbf{2}^{(\omega)}$; that is, $[x] = x + \mathbf{2}^{(\omega)}$.

6 Formal proof of the finite Boolean representation theorem

I gave an informal proof of the finite Boolean representation theorem above. [4/ 10.1 above] Now I proceed to a formal proof. We have seen that a Boolean algebra is atomic iff $\bigvee_{x \in A} x = 1$.

6.1 Definition, set of all atoms

Let B be a Boolean algebra and let $x \in B$. Then we define $\Psi(x)$ to be the set of all atoms $b \in B$ such that $b \leq x$ and $A = \Psi(1) =$ the set of all atoms of B. That is

$$\Psi \begin{cases} B \to A = \Psi(1) \\ x \to \{b_i\} \text{ where } b_i \text{ is an atom and } b_i \leq x \end{cases}$$

This is a contraction mapping of the lattice onto the set of atoms, which is its skeleton. We also have $\Psi(\mathbf{0}) = \mathbf{0}$. $\mathbf{0}$ is the fixed point of the contraction mapping. What this is saying is that every element $x \in B$ can be uniquely written as the join of the atoms that it dominates: $x = \bigvee_{\alpha \in \Psi(x)} \alpha$. To confirm this we need the following results.

6.2 Lemma, isomorphism within of Boolean algebra of the sets of atoms

Let B be a Boolean algebra with $\Psi(x)$ defined as above. Then the function $\Psi(x)$ is one-one: $\Psi(x) = \Psi(y)$ iff $x = y$.

<u>Proof</u>

Suppose $x \neq y$. Then either $x \not\leq y$ or $y \not\leq x$. Let us assume $x \not\leq y$. Then $x \wedge y' \neq \mathbf{0}$. The fact that B is atomic implies that there exists an atom $\alpha \leq x \wedge y'$. This implies $\alpha \leq x$ ad $\alpha \in \Psi(x)$. But it also implies $\alpha \leq y'$, hence $\alpha \notin \Psi(y)$. Hence $\Psi(x) \neq \Psi(y)$.

6.3 Theorem, isomorphism theorem

Let B be a Boolean algebra, let Ψ be the isomorphism from $x \in B$ to the set of all atoms $b \leq x$ as above, and let $A = \Psi(1)$. Then, Ψ is an isomorphism from B into $\mathbf{P}(A)$.

<u>Prerequisites</u>

1. The power set of a set $X = \{\alpha_i\}$ is the closure under the operations of relative complementation and intersection of the sets $\{\alpha_1\}, \{\alpha_2\}, \ldots$
2. The Boolean algebra B is closed under the operations of complementation and join. That is, if $x \in B$ then $x' \in B$ and if $x, y \in B$ then $x \wedge y \in B$.
3. The lemma above, that $\Psi(x)$ is one-one from B into the set $\mathbf{P}(a)$ of all subsets of $A = \Psi(1)$.

<u>The idea of the proof</u>

Since Ψ is already shown to be one-one we have to show that it is into $\mathbf{P}(A)$. Since the set of all atoms relative to an element x may be different relative to its complement x' we have to show that the sets $\Psi(x)$ and $\Psi(x')$ partition the set of atoms A. Likewise, we have to show that $\Psi(x), \Psi(y) \in \mathbf{P}(A)$ imply $\Psi(x \wedge y) \in \mathbf{P}(A)$.

Proof

1. Possibly, $\Psi(x') \notin \mathbf{P}(A)$. But we can show that $\Psi(x') = (\Psi(x))'$. For let b be an atom. Then $b \le x'$ implies $b \not\le x$. Therefore, $\Psi(x') \subseteq (\Psi(x))'$. Likewise $b \not\le x$ implies $b \le x'$ and $(\Psi(x))' \subseteq \Psi(x')$. What this says is that $\Psi(x)$ and $\Psi(x')$ are disjoint sets that partition $A = \Psi(1)$.

   So $\Psi(x') \subseteq \mathbf{P}(A)$,

2. Possibly $\Psi(x \wedge y) \notin P(A)$. But we can show

   $\Psi(x \wedge y) = \Psi(x) \cap \Psi(y)$.

   For let b be an atom. Then $b \le x \wedge y \Rightarrow b \le x$ and $b \le y$ so

   $b \in \Psi(x \wedge y) \Rightarrow b \in \Psi(x)$ and $b \in \Psi(y) \Rightarrow b \in \Psi(x) \cap \Psi(y)$

   Likewise,

   $b \in \Psi(x)$ and $b \in \Psi(y) \Rightarrow b \le x \wedge y$

   $b \in \Psi(x) \cap \Psi(y) \Rightarrow b \in \Psi(x \wedge y)$

   So $\Psi(x \wedge y) \in \mathbf{P}(A)$.

What this theorem affirms that a lattice point, say $p \vee q$, maps to a subset of the atoms; it is uniquely represented by a subset of atoms. For example, in $\mathbf{2}^4$ we have $p \vee q \leftrightarrow \{1,2,3\}$ or equivalent. However, the theorem is valid for all *atomic* Boolean algebras, both finite and infinite. I state the following theorem without proof.

6.4 Theorem, size of finite a Boolean algebra

Let *B* be a Boolean algebra, and let *n*, *n* finite, be the number of elements in the set $A = \Psi(1)$ of all atoms of *B*. That is $n = |\Psi(1)|$. Then $|B| = 2^n$.[18]

This result, together with the main theorem above, yields the finite Boolean representation theorem.

6.5 Corollary, the finite Boolean representation theorem

If *B* is finite, then this isomorphism Ψ is onto (and hence a bijection).

It is important to appreciate the distinction here between "into" and "onto". The isomorphism theorem above yields the result that any atomic Boolean algebra, *B*, is contained within the power set of its atoms but it does not equate that Boolean algebra with its power set. In fact,

[18] A proof may be found in Mendelson [1970] p. 186.

it does not even say that *B* is isomorphic to a field of sets contained within that power set. For the finite case, atomic Boolean algebra, complete field of sets and power set of the atoms all coincide. This result significantly does not extend to the infinite case.

7 Ideals in a Boolean algebra

Recall the fundamental definitions for a lattice.

JOIN $x \vee y = \sup(x,y) \in L$ MEET $x \wedge y = \inf(x,y) \in L$

Filters (up-sets) and ideals (down-sets) were introduced earlier [4.6.1]. Filters are very important for analytic logic built over a lattice since the filter of a lattice point ϕ is synonymous to the consequences of that lattice point: $\phi \vDash \dots$. Filters and ideals are dual concepts, and in the context of Boolean algebras it is usual to develop the theory primarily for ideals and argue by duality that the same theory applies to filters.

7.1 Definition, ideal

Let $B = (B, \wedge, \vee, ', 0, 1)$ be a Boolean algebra. Let $J \subseteq B$ be a non-empty subset of *B*.

The *J* is said to be an *ideal* of *B* if

1. $x, y \in J \Rightarrow x \vee y \in J$
2. $x \in J, b \in B \Rightarrow x \wedge b \in J$

Examples

$\{0\}$ and *B* are both ideals of a Boolean algebra *B*.

7.2 Definition, proper ideal

Every ideal of a Boolean algebra *B*, different from *B* is said to be a *proper ideal* of *B*.

7.3 Result

Let *J* be an ideal of a Boolean algebra *B*, let $x \in J$ and $y \le x$, then $y \in J$.

What this result emphasises is that everything that lies below a point $x \in J$ in an ideal also lies in the ideal *J*, and this is why they are called *down sets*. This also means that the natural way to picture an ideal is as the down set of some element of the lattice $x \in J$, so that the ideal is generated from the top down. As it happens, not all ideals can be generated in this way. When they are, they are called *principal ideals*.

7.4 Definition, principal ideal

$J_u = \{v \in B : v \le u\}$ in the above result is said to be a *principal ideal* of *B*.

Examples

1. $\{0\}$ is a proper ideal .
2. J is a proper ideal iff $1 \notin J$.

Principal ideals are defined by "topmost elements". To say that an ideal is principal is to say that there is a "topmost" that is principal element in the lattice and that the ideal comprises every other element in the lattice that lies below this. That is, an ideal J is principal in Boolean algebra B iff $(\exists u)\left(J = J_u = \{v \in B : v \le u\}\right)$. (The term "topmost" is not standard and is introduced here to help visualise the concept.)

7.5 Result, principal ideal

Let $u \in B$ be an element of a Boolean algebra B. Then the set $J_u = \{v \in B: v \le u\}$ is an ideal of B. (Proof, Mendelson [1970] p. 144)

In a finite Boolean algebra every ideal must be principal. This follows immediately from the definition of an ideal where we have $x, y \in J \Rightarrow x \vee y \in J$ and in a finite Boolean algebra every set of lattice points has a join.

7.6 Properties of principal ideals

(Mendelson [1970] examples 5.9,5.10)

1. Let A be a non-empty set and $\mathbf{P}(A)$ the Boolean algebra on A. Then the atoms of $\mathbf{P}(A)$ are the singleton sets $\{a\}$ where $a \in A$. Hence any maximal principal ideal consists of all subsets X of A such that $a \notin X$.

   Example

   $\mathbf{2}^4$ is the algebra of subsets of $A = \{1,2,3,4\}$. The atoms are $\{1\},\{2\},\{3\},\{4\}$ and by deleting any one member of A, we obtain a maximal principal ideal. These are $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$.
2. In any finite Boolean algebra every maximal ideal is a principal ideal.

In an infinite Boolean algebra not every infinite set of lattice points need have a join, or alternatively, that join need not be contained in the ideal. So to consider non-principal ideals we must be discussing an infinite lattice. The paradigm of a non-principal idea is the ideal of all finite subsets of ω, which is a subset of the Cantor set, $\mathbf{2}^\omega \cong \{0,1\}^\omega$. We can see automatically that this subset is (a) an ideal and (b) cannot be principal. It is an ideal because for every two finite subsets of ω there is a join; it cannot be principal because there cannot be a single finite subset of ω that contains every other finite subset of ω. This is a consequence of the manner in which the ideal is generated *from the bottom upwards* as a *potentially infinite structure*. As it follows that in any infinite set not all ideals are principal it also follows that there must be some other method of generating an ideal – one that proceeds from the "bottom up" just as these remarks suggest.

7.7 Result, generator

Let $X \subseteq B$ be a subset of a Boolean algebra *B*. Then the intersection $J = \bigcap J_i$ such that J_i is an ideal containing *X*, $(X \subseteq J_i)$, is itself an ideal such that $X \subseteq J$.

7.8 Definition, generator

The set *X* corresponding to the ideal *J* in the above result is said to be the *generator* of *J*. We write this $J = \text{Gen}(X)$.

7.9 Theorem

Let *X* be any subset of a Boolean algebra, *B*, then $\text{Gen}(X)$ is the set

$$\text{Gen}(X) = \{(x_1 \wedge b_1) \vee ... \vee (x_i \wedge b_i) \vee ... \vee (x_k \wedge b_k) : x_i \in X \text{ and } b_i \in B\}.$$

(Proof, Mendelson [1970] p. 144)

7.10 Theorem

Let *X* be any subset of a Boolean algebra, *B*, then $\text{Gen}(X)$ is the set

$$\text{Gen}(X) = \{y \leq x_1 \vee ... \vee x_i \vee ... \vee x_k : x_i \in X\}$$

(Proof, Mendelson [1970] p. 144)

To say that an ideal is principal in a Boolean algebra: $(\exists u)(J = J_u = \{v \in B : v \leq u\})$ means that this principal element *u* is the disjunction of elements of a set *X* that generates *J*. In other words, to say an ideal *J* is principal is to say that it has a set *X* that generates *J* such that $(\exists X)(J = \text{Gen}(X))$ and $u = \bigvee_{x \in X} x$, $J = J_u$. This is the result that clarifies the meaning of generators in this context. The generators could be atoms, if atoms exist. If atoms do not exist, take any subset *X* and form the disjunction of its elements $x_1 \vee ... \vee x_i \vee ... \vee x_k : x_i \in X$. Then $\text{Gen}(X)$ is the ideal that lies below this topmost element

7.11 Result

Let *X* be any subset of a Boolean algebra, *B*, then $\text{Gen}(X)$ is proper ideal of *B* iff

$x_1 \vee ... \vee x_i \vee ... \vee x_k \neq 1$ where $x_i \in X$.

(Proof Mendelson [1970] p. 144)

7.12 Result

(Mendelson [1970] p. 160) Let B be an atomic Boolean algebra. Then very element $x \in B$ is the l.u.b. (supremum) of the set: -

$\Psi(x) = \{b \in B : b \text{ is an atom } \& \ b \leq x\}$ and x is not the l.u.b. of any proper subset of $\Psi(x)$.

Proof

We have $\Psi(x) = \{b \in B : b \text{ is an atom } \& \ b \leq x\}$, so the condition $b \leq x$ implies x is an upper bound for $\Psi(x)$.

Let z be an upper bound of $\Psi(x)$ such that $x \not\leq z$.

Then $x \not\leq z \Rightarrow x \wedge z' \neq 0$.

Since B is atomic there exists an atom $b \leq x \wedge z'$.

Therefore, $b \leq x$ and $b \in \Psi(x)$.

Also $b \leq z'$. But as z is an upper bound for $\Psi(x)$, then $b \leq z$. Therefore, $b \leq z \wedge z' = 0$ which is a contradiction, since b is an atom. Therefore, z must be an upper bound of $\Psi(x)$ such that $x < z$, an so x must be the l.u.b. of $\Psi(x)$. (For the second half of this proof, Mendelson [1970] p. 160)

What this means is that every set of atoms generates an ideal. Ideals are generalised Boolean algebras embedded in a larger Boolean algebra.

7.13 Definition, subalgebra

(Givant and Halmos [2009] p. 74)

A Boolean subalgebra of a Boolean algebra A is a subset B of A such that B together with the distinguished elements and operations of A (restricted to the set B) is a Boolean algebra. The algebra A is called a Boolean extension of B.

7.14 Definition, generalised Boolean algebra

(Birkhoff [1940] who attributes this to Stone)

A generalised Boolean algebra is an algebra that has no largest element **1**, and hence is not a complemented lattice. However, it is a relatively complemented, distributive lattice. [See 5.8 above]

Thus, we have ideals that are not principal. This, therefore, invites the following definition: -

7.15 Definition, maximal ideal

An ideal M of a Boolean algebra B is said to be *maximal* if M is a proper ideal and if there is no proper ideal J of B such that $M \subset J$.

This definition is essential in order to distinguish two concepts of "maximality" – the first is the natural concept of principal ideal where the ideal may be thought of as generated top down from a "topmost" element; the second arises from the bottom up approach as a species of limit – it is the largest ideal that can be generated from the bottom up without reaching the topmost element of the lattice as a whole, that is **1**. If the lattice is atomic then this makes the distance of 1 above M just 1. The distance from an atom to **0** is also just 1. This last remark makes it clear that *maximal ideals are to* **1** in the lattice as *atoms are to* **0**.

It follows from the definition of an ideal that $\mathbf{2}^{<\omega}$ is an ideal in $\mathbf{2}^{\omega}$, but we *have not proven* that $\mathbf{2}^{<\omega}$ is a maximal ideal. It is known that it is *impossible* to do this in ZF set theory, and that its proof requires the Axiom of Choice [3.12 above], specifically in its equivalent form of Zorn's lemma: -

7.16 Zorn's lemma

Let X be a partially ordered set in which every chain (i.e totally ordered subset) has an upper bound, then X possesses a maximal element.

7.17 Definition, subset chain

Let S be a set of sets. Then a $\subset$-chain in S is a subset of S such that if $A \in C$ and $B \in C$ where $A \neq B$, then either $A \subset B$ or $B \subset A$.

More generally

Let R be a binary relation on a set W. Then an R-chain in W is a subset of W on which R is transitive, connected and antisymmetric.

7.18 Definition, technical variant of Zorn's lemma

Let S be a set of sets such that for every $\subset$-chain C in S, the union $\bigcup_{A \in C} A$ is also in S. Then there is a C-maximal set M in S.

7.19 Definition, prime ideal

Let J be an ideal of a Boolean algebra B. Then J is said to be *prime* iff for all $x, y \in B$ such that $x \notin J$ and $y \notin J$ we have $x \wedge y \notin J$.

7.20 Theorem

A proper ideal J in a Boolean algebra B is maximal iff it is Boolean prime ideal.
(Proof, Mendelson [1970] p. 145)

So there is some duplication of terminology here since prime ideal means the same as maximal proper ideal. This means that in an infinite Boolean algebra, prime = maximal. Not all maximal ideals are principal: the join of an infinite set does not necessary exist in the lattice.

7.21 Maximal ideal theorem

Let *J* be a proper ideal of a Boolean algebra *B*. Then there is a maximal ideal *M* in *B* such that $J \subseteq B$. This means that every proper ideal can be extended to a maximal ideal. (Note underlining.)

Proof

Let *J* be a proper ideal of a Boolean algebra *B*. Let *Z* be the class of all proper ideals *K* such that $J \subseteq K$. Let *C* be any $\subset$-chain in *Z*. Then $\bigcup_{I \in C} I$ is a proper ideal containing *J*. However, this is automatic, because by definition every element in *M* contains *J*, so the union of all such ideals in *C* must contain *J*. Hence, by Zorn's lemma there is a maximal set *M* in *Z* such that $J \subseteq M$. Furthermore, *M* must be a maximal ideal in *B*. To show this, let M^* be a proper ideal such that $M \subseteq M^*$. Then $M^* \in Z$ so $M^* = M$.

7.22 Corollary

Let $2^{\omega} = P(\omega)$ be the Boolean algebra of all subsets of the infinite set ω. Let $F(\mathbb{N})$ be the ideal of all finite subsets of ω. Then $F(\mathbb{N})$ can be extended to a maximal ideal M.

7.23 (+) The Maximal idea theorem is the central result of representation theory

This is the central result of the representation theory of infinite Boolean lattices. The actual representation theorem follows below [7.29 below], and is also called the Prime Ideal theorem. In essence, these are all the same theorem and are consequences of the application of the Axiom of Choice to the ideal $F(\mathbb{N})$ to extend it to a maximal ideal M. It is this result that justifies the distinction we have drawn between the two concepts, and also indicates that they are separate structures.

I have taken the statement of this theorem from Mendelson [1970]. However, the expression "in *B*" in it (underlined above) [Second line 7.21] is ambiguous. $F(\mathbb{N})$ is an ideal in *B* (which is here $B = \mathbf{2}^{\omega}$) but its maximal ideal, M, in effect demonstrates that *B* (where $B = \mathbf{2}^{\omega}$) is a complete Boolean algebra in its own right. $FC(\mathbb{N}_{\infty})$ is a Boolean algebra of which $F(\mathbb{N})$ is an ideal, but M is not a maximal ideal of $FC(\mathbb{N}_{\infty})$; M belongs to the Cantor space, $\mathbf{2}^{\omega}$ of which $FC(\mathbb{N}_{\infty})$ is a sub-algebra. Hence, the Axiom of Choice is allowing us to extend and complete $FC(\mathbb{N}_{\infty})$ by embedding it in $\mathbf{2}^{\omega}$.

We have $F(\mathbb{N}) = \mathbf{P}(0) \subset \mathbf{P}(1) \subset \mathbf{P}(2) \subset \ldots$ as an unbounded sequence. Just like $\mathbb{N}$ it has no maximal element and satisfies the Archimedean property [See 3.1 above]. And yet, by applying Zorn's lemma to it, we have demonstrated the existence of $\mathrm{M} = \max\{F(\mathbb{N})\}$. Compare with $\mathbb{N} \subset \mathbb{N}_\infty = \{0,1,2,\ldots,\infty\}$, so that $\max(\mathbb{N}) = \infty$. We have $\infty = \max(\mathbb{N}) \notin \mathbb{N}$, so that $\mathbb{N}$ remains unbounded above, but $\infty = \max(\mathbb{N}) \in \mathbb{N}_\infty$, so that it does have a maximum. By a similar argument there is a sequence of ideals $\mathcal{M} = \{m_0, m_1, m_2, \ldots, \mathrm{M}\}$ of which M is the last with $F(\mathbb{N}) \cong \{m_0, m_1, m_2, \ldots\}$ as an unbounded sequence, so that $\mathrm{M} \notin F(\mathbb{N})$ yet $\mathrm{M} \in \mathcal{M}$.

Under the hypothesis $\omega = \mathbb{N}$ the application of Zorn's lemma in the Maximal ideal theorem leads to an actual contradiction since it ascribes a maximum to a sequence $F(\mathbb{N})$ that could not have one. The "paradox" is resolved by $\mathbb{N}_\infty \neq \mathbb{N}$ $(\omega \neq \mathbb{N})$ so the sequence on which Zorn's lemma is applied in this case is not $F(\mathbb{N})$ but $\mathcal{M} = \{m_0, m_1, m_2, \ldots, \mathrm{M}\} = F(\mathbb{N}) \cup \mathrm{M}$.

Zorn's lemma is also used in the construction of the one-point compactification of $\mathbb{N}$ [Result 3.22 above], and it seems that its function (as with its equivalent Axiom of Choice) is to assert the existence of an actually infinite complete partition of the interval.

7.24 Result

Every proper ideal *J* of a Boolean algebra *B* is equal to the intersection *H* of all maximal ideals containing it. (Proof Mendelson [1970] 5.34)

This means that every proper ideal can be extended to a maximal ideal. Given the equivalence: maximal proper ideal $\equiv$ prime ideal, this is equivalent to: -

7.25 Prime ideal theorem

Let *J* be a proper ideal of a Boolean algebra *B*. Then there is prime ideal in *B* that contains *J*.

7.26 Lemma, constructing ideals by adjoining elements

1. Let J be an ideal of a Boolean algebra B and let $b \in B$. Then $\mathrm{Gen}(J \cup \{b\})$ is an ideal consisting of all elements of the form $(y \wedge b) \vee x$ where $y \in B$ and $x \in J$. (Proof Mendelson [1970] p.145)
2. Let *J* be an ideal of a Boolean algebra *B* and let $b \in B - J$. Then the ideal $\mathrm{Gen}(J \cup \{b\})$ is a proper ideal iff $x \vee b \neq 1$ for all $x \in J$. That is, for all $x \in J$, iff $b' \not\leq x$. (Proof Mendelson [1970] p.145)

This is a kind of technical lemma required in the proof of the theorem below: -

7.27 Theorem, maximal ideals as partitions

Let *M* be a proper ideal of a Boolean algebra, *B*. Then *M* is maximal iff for any $b \in B$ either $b \in M$ or $b' \in M$.

1. Suppose *M* is maximal and $b \notin M$. Let $J = \text{Gen}(M \cup \{b\})$. This defines an ideal such that $M \subset J$. Since *M* is maximal, this entails $J = B$. Hence by the lemma above $b' \leq x$ for some $x \in M$. Since *M* is an ideal this entails $b' \in M$.
2. Conversely, suppose either $b \in M$ or $b' \in M$ for all $b \in B$. Let $M \subset J$ where *J* is an ideal. Suppose $y \in J - M$. This means $y \in J$ and $y \notin M$, hence $y' \in M$. Then $y \vee y' = 1 \in J$. Hence $J = B$, which entails that *M* is maximal.

7.28 Corollary

(Givant and Halmos [2009] p. 173, also Mendelson [1970] p.149)

Let *J* be any proper ideal of a Boolean algebra *B*. For every element *p* in *B* that does not belong to *J* there exists a maximal ideal *M* that includes *J* but does not contain *p*.

<u>Proof</u>

The ideal *N* generated by $J \cup \{p'\}$ is proper. Then the maximal ideal theorem entails that there is a maximal ideal that contains *N* and hence also *M*. As it is a maximal ideal that contains p' it cannot contain *p*.

7.29 Corollary, Boolean representation theorem (Birkhoff)

Every Boolean algebra is isomorphic to a field of sets.

<u>Proof idea</u>

For any Boolean algebra *B* there is an isomorphism into the power set $\mathbf{P}(M)$ of the set of all maximal ideals **M** in *B*. The image of this isomorphism is a field of sets contained in $\mathbf{P}(M)$.

<u>Proof</u>

Let *B* be any Boolean algebra. Let *m* denote a maximal ideal. Let *M* denote the set of all maximal ideals. For $x \in B$ let

$$\begin{aligned} \mathbf{X}(x) &= \{\text{all maximal ideals } m \in M \text{ such that } x \notin m\} \\ &= \{m \in M : x \notin m\} \\ &= \{m \in M : x' \in m\} \quad \text{since for any } x \in B \text{ either } x \in m \text{ or } x' \in m \end{aligned}$$

Then $\mathbf{X}(1)$ is the set of all maximal ideals of *B* and $M = \mathbf{X}(1)$. We also have

1. $\mathbf{X}(0) = \varnothing$.
2. $x \neq 0 \Rightarrow \mathbf{X}(x) \neq \varnothing$.

3. $\mathbf{X}(x') = (\mathbf{X}(x))'$ as every maximal ideal contains either x or x'.

4. $\mathbf{X}(x \vee y) = \mathbf{X}(x) \cup \mathbf{X}(y)$

Let m denote a maximal ideal. That is $m \in M$.

$x \notin m \Rightarrow x \vee y \notin m$ since $x \le x \vee y$. Similarly, $y \notin m \Rightarrow x \vee y \notin m$ since $y \le x \vee y$. Hence, $\mathbf{X}(x) \cup \mathbf{X}(y) \subseteq \mathbf{X}(x \vee y)$. Conversely

$x \in m$ and $y \in m \Rightarrow x \vee y \in m$. Thus, by contraposition

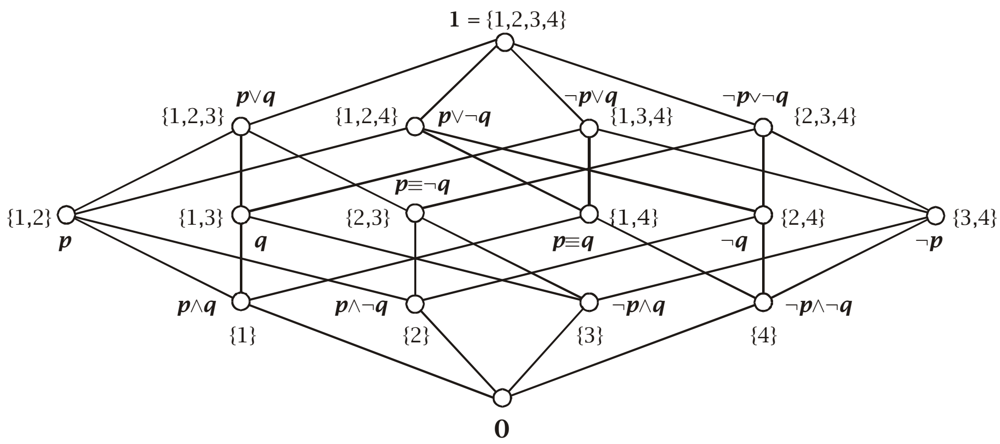
$x \vee y \notin m \Rightarrow x \notin m$ or $y \notin m$. Hence

$\mathbf{X}(x \vee y) \subseteq \mathbf{X}(x) \cup \mathbf{X}(y)$.

Hence $\mathbf{X}$ is an isomorphism from B into $\mathbf{P}(M)$, the set of all subsets of the set M of all maximal ideals.

7.30 Finite example

The Boolean algebra $\mathbf{2}^4$: -



The maximal ideals (Boolean primes) are

$m_1 = \{p \in \mathbf{2}^4 : 1 \notin p\} = \{\{2,3,4\},\{2,3\},\{2,4\},\{3,4\},\{2\},\{3\},\{4\},\varnothing\}$

$m_2 = \{p \in \mathbf{2}^4 : 2 \notin p\} = \{\{1,3,4\},\{1,3\},\{1,4\},\{3,4\},\{1\},\{3\},\{4\},\varnothing\}$

$m_3 = \{p \in \mathbf{2}^4 : 3 \notin p\} = \{\{1,2,4\},\{1,2\},\{1,4\},\{2,4\},\{1\},\{2\},\{4\},\varnothing\}$

$m_4 = \{p \in \mathbf{2}^4 : 4 \notin p\} = \{\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\varnothing\}$

$\mathbf{X}(x) = \{$all maximal ideals $m \in M$ such that $x \notin m\}$

$= \{m \in M : x \notin m\}$

$= \{m \in M : x' \in m\}$ since for any $x \in B$ either $x \in m$ or $x' \in m$

$\mathbf{X}1 = \mathbf{X}(\{1\}) = \{m_1\}$

$\mathbf{X}1 = \mathbf{X}(\{2\}) = \{m_2\}$

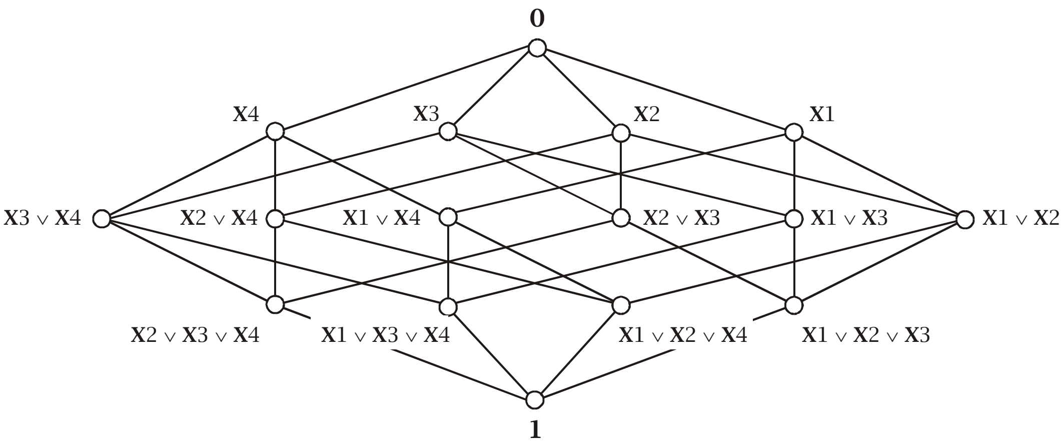
$\mathbf{X}1 = \mathbf{X}(\{3\}) = \{m_3\}$

$\mathbf{X}1 = \mathbf{X}(\{4\}) = \{m_4\}$

$$\begin{aligned}\mathbf{X}1 \vee \mathbf{X}2 &= \mathbf{X}(\{1\}) \vee \mathbf{X}(\{2\}) = \mathbf{X}(\{1\}) \cup \mathbf{X}(\{2\}) = \{m_1\} \cup \{m_2\} = \{m_1, m_2\}\\ &= \mathbf{X}(\{1,2\}) = \mathbf{X}(\{1\} \vee \{2\})\\ &= \{\text{all maximal ideals } m \in M \text{ such that } \{1,2\} \notin m\}\\ &= \{m \in M : \{1,2\} \notin m\}\\ &= \{\mathbf{X}(\{1\}), \mathbf{X}(\{2\})\} = \{\mathbf{X}1, \mathbf{X}2\}\end{aligned}$$

Likewise

$$\mathbf{X}1 \vee \mathbf{X}2 \vee \mathbf{X}3 = \{\mathbf{X}(\{1\}), \mathbf{X}(\{2\}), \mathbf{X}(\{3\})\}$$



Compare this with

$$\begin{aligned}m_1 \cup m_2 &= \{\{1,3,4\},\{2,3,4\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1\},\{2\},\{3\},\{4\},\varnothing\}\\ &= \{\{1,2,3\},\{1,2,4\}\}'\end{aligned}$$

and

$$m_1 \cap m_2 = \{\{3,4\},\{3\},\{4\},\varnothing\}$$

All of these express the same underlying facts.

For the finite case we generate the Boolean algebra upwards as the power set of the set of its atoms, and the construction involving maximal ideals is not necessary. For infinite cases this is not possible. Therefore, we *assume* that the corresponding maximal ideals exist, which we justify by the Axiom of Choice, and generate the complete Boolean algebra downwards as a algebra of maximal ideals. Each maximal ideal defines an atom of this dual algebra for which the set of atoms is a basis.

7.31 Result

Every proper ideal J of a Boolean algebra B is equal to the intersection H of all maximal ideals containing it.

Proof

By the maximal ideal theorem there is a maximal ideal containing *J*. Then $J \subseteq H$. To show that $H \subseteq J$, let $x \notin J$. By the lemma on constructing ideals by adjoining elements, $J \cup \{x'\}$ is a proper ideal. By the maximal ideal theorem, there is a maximal ideal M such that $\text{Gen}(J \cup \{x'\}) \subseteq M$. Whence $J \subseteq M$ and $x \notin M$. But H is the intersection of all maximal ideals containing *J*, hence $x \notin H$ and $H \subseteq J$.

7.32 Result

A Boolean algebra *B* is isomorphic to a field $\mathbf{P}(X)$ of all subsets of a non-empty set *X* iff *B* is complete and completely distributive. (Proof Mendelson [1970] p.174)

7.33 Alternative approaches to the Boolean representation theorem

The approach here is to derive the Boolean representation theorem by the following route: (1) Definition of maximal ideals; (2) Zorn's lemma to establish the maximal ideal theorem; (3) Maximal ideals acting as a basis for the Boolean algebra taking the place of atoms; (4) Boolean representation theorem as equating the Boolean algebra in the field of sets generated by that basis.

It is as well to note that there is an alternative approach to this theorem, which is one that might also be more appropriate to a universal algebra[19]. (1) Definition of homomorphisms; (2) Definition of quotient algebras; (3) Decomposition of algebras into factors of indecomposable algebras; (4) Use of Zorn's lemma to establish Birkhoff's theorem: Every algebra *A* is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of *A*.

The first approach is due to Stone and the second to Birkhoff. The two approaches are equivalent.

7.34 Outline of the "universal" approach to the Boolean / Stone representation theorem

An algebra *A* is *directly indecomposable* if *A* is not isomorphic to a direct product of two nontrivial algebras.

7.35 Theorem

Every finite algebra is isomorphic to a direct product of directly indecomposable algebras.

[19] For this approach see also Burris and Sankappanavar [1981]

For Boolean algebras the only directly indecomposable algebra is $\mathbf{2} = \{0,1\}$. So all Boolean algebras are built up as products of this unit. Although every finite algebra is isomorphic to a direct product of directly indecomposable algebras, the same does not hold for infinite algebras in general. To find building blocks for algebras we therefore require the notion of a subdirect product.

7.36 Definition, subdirect product

An algebra A is a subdirect product of an indexed family $(A_i)_{i\in I}$ of algebras if

(i) $A \leq \prod_{i\in I} A_i$

(ii) $\pi_i(A) = A_i$ for each $i \in I$.

7.37 (+) Result

The Cantor set $\mathbf{2}^{\omega}$ is the direct product of ω copies of $\mathbf{2} = \{0,1\}$, but $\mathbf{2}^{<\omega}$, the Boolean algebra of all finite subsets of $\mathbb{N}$, is not the direct product of any denumerable number of copies of **2**. It is contained in $\mathbf{2}^{\omega}$ as a subfield.

Proof

Suppose $\mathbf{2}^{<\omega}$ is the direct product of copies of **2**. Either $\mathbf{2}^{<\omega}$ is the direct product of a finite copies of **2** or of infinite. If infinite, then $\mathbf{2}^{<\omega} \cong \mathbf{2}^{\omega}$ which is false, because $\mathbf{2}^{<\omega}$ is a proper subset of $\mathbf{2}^{\omega}$. If finite, then $|\mathbf{2}^{<\omega}|$ is finite, which is too small.

As an algebra $\mathbf{2}^{<\omega}$ lies somewhere between $\mathbf{2}^{k}$ where k is *any* integer and $\mathbf{2}^{\omega}$. This makes it the subdirect product of **2**.

7.38 Theorem (Birkhoff)

Every algebra A is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of A.

Notes

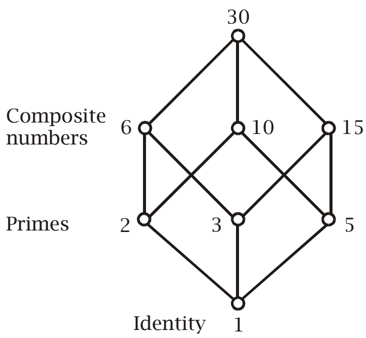
The proof of this requires Zorn's lemma.

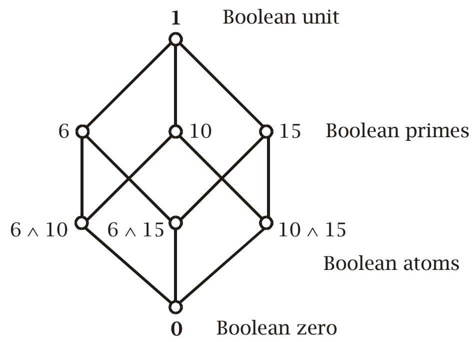
7.39 Representation of Boolean prime ideals in the lattice of prime divisors

The definition of an ideal in the language of a Boolean ring becomes, $x, y \notin J \Rightarrow x \cdot y \notin J$. This makes membership of an ideal appear as a divisibility property. By contraposition $x \cdot y \in J \Rightarrow x, y \in J$. A Boolean prime is equivalent to a composite number that has a unique prime factorisation. The duality principle [4.2.5] allows for the interchange of ideals for filters; this means that to each Boolean prime *ideal* there is a Boolean prime *filter*. In the lattice of divisors, these Boolean prime filters actually are prime numbers.

7.40 Example

As the following finite example illustrates, this reverses the usual definition of prime and composite.





7.41 Elements in a complete Boolean algebra

The combination of (1) $\omega \neq \mathbb{N}$, (2) the Axiom of choice making $\mathbb{N}$ and μ into sets, (3) the extension of the power set operation to these results, (4) the definition of $\mathrm{M} = \max\{F(\mathbb{N})\}$ and (5) $F(\mathbb{N}) = \mathbf{P}(0) \subset \mathbf{P}(1) \subset \mathbf{P}(2) \subset \ldots$ together justify the assertion $\mathrm{M} = \mathbf{P}(\mu)$ $(\mathrm{M} = \mathbf{P}(\mathbb{N}))$. The status of μ in this needs to be clarified. We have $\mathbb{N}_\infty = \mu \cup \{\infty\}$ $(\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\})$, which indicates that μ is not an element or atom of the partition of $[0,1]$ but a collection of them: $\mu = \{0\} \cup \{1\} \cup \{2\} \cup \ldots$; μ is not a lattice point of $F(\mathbb{N})$. Since the complete algebra $\mathbf{2}^{\mathbb{N}_\infty}$ contains all infinite meets and joins, $\mu \in \mathbf{2}^{\mathbb{N}_\infty}$ is a lattice point of this algebra, just as all its subsets are.

7.42 (+) Definition, maximal "element"

$F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ has no maximum in itself. It is not a principal ideal in $\mathbf{2}^{\omega}$. This is because $\mathbb{N}$ has no maximum – it is unbounded. However, within $\mathbf{2}^{\omega}$, $F(\mathbb{N})$ is bounded above, and we say that $\mathrm{M} = \max\{\mathbf{2}^n\}$ is its maximal ideal; we define μ to be such that $\mathrm{M} = \mathbf{P}(\mu)$ and say that it is its *maximal element*.

An incomplete algebra is one in which not all infinite meets and joins exist. Hence, there are ideals in such an algebra that are not principal. When the algebra is extended to a complete algebra *X*, in *X* all ideals are principal. Hence a maximal element is a principal element that is a member of a complete algebra but not a member of its incomplete embedding.

This result is consistent with Rubin's proposition [3.18 above] – yet another version of the Axiom of Choice – which states that the Axiom of Choice is equivalent to the statement, "The power set of every ordinal is well-orderable"; μ is not an ordinal, but under the Axiom of Choice it becomes an ordered set, and hence similar [Defined, Chap.2 / 2.8.3] to an ordinal; then $\mathrm{M} = \mathbf{P}(\mu)$ is also well-ordered.

The role of the maximal "element" is analogous to the role played by a *real number* in its relation to an infinite sequence of *rational numbers* as in the Dedekind cut or any other definition of completeness. Just as $\sqrt{2}$ is *not* an element of the sequence of rational numbers that defines it, so too the maximal "element" may or may not be an element of the ideal of which it is the supremum. When it is not, I shall also call such an element *a boundary element*. In the paradigmatic case of the ideal of all finite subsets of some infinite set *X*, the maximal element belongs to the boundary between the finite and infinite in the larger, atomic lattice $\mathbf{P}(X)$.

CHAPTER 6

Cantor space

1 Boolean and Stone spaces

1.1 Definition, Boolean space

A topological space that is compact and totally disconnected is said to be a *Boolean space.*

1.2 Theorem, existence of Stone space

Let M be the set of maximal ideals of a Boolean algebra B. For any $x \in B$ define $\mathbf{X}(x) = \{m \in M : x \notin m\}$. Define a topology T for M so that arbitrary unions of sets of the form $\mathbf{X}(x)$ are open sets of T. Then $S_B = (M,T)$ is a Boolean space in which the clopen sets of M are the sets $\mathbf{X}(x)$. $S_B = (M,T)$ is called the *Stone space* of B.

Example

[See example 5 / 7.30] Before proving this, let us illustrate it with a finite example. In $\mathbf{2}^4$ the prime ideals (maximal ideals) are given by

$\mathbf{X}(\mathbf{1}) = \{m \in M : \mathbf{0} \in m\} = \{\mathbf{X}(\{1\}), \mathbf{X}(\{2\}), \mathbf{X}(\{3\}), \mathbf{X}(\{4\})\} = \{m_1, m_2, m_3, m_4\}$

The topology T on M generates the Stone space where the prime ideals, m_1, m_2, m_3, m_4 of $\mathbf{2}^4$ become its atoms. The atoms constitute a partition of the Stone space, which is compact and totally disconnected. There is a Boolean algebra which is the power set $\mathbf{P}(\mathbf{X}(\mathbf{1})) = \mathbf{P}(\{m_1, m_2, m_3, m_4\})$ which is isomorphic to $\mathbf{2}^4$, the original Boolean algebra. The Stone space is isomorphic to $\mathbf{2}^4$ but is an inverted copy of it.

Proof of the theorem

If m is a maximal ideal then m is a proper ideal; hence $m \in \mathbf{X}(x)$ for some $x \in B$. To show that $S_B = (M,T)$ is a Boolean space we must show that it is totally disconnected and compact.

1. A space S is totally disconnected iff for distinct points $a, b \in S$ there exists a clopen set $C \subseteq S$ such that $a \in C$ and $b \notin C$. Now suppose that $m_1, m_2 \in M$ are distinct maximal ideals. Then there is a $x \in m_1 - m_2$

where $m_1 \in \mathbf{X}(x')$ and $m_2 \in \mathbf{X}(x)$. Since $\mathbf{X}(x') = [\mathbf{X}(x)]'$ we have $m_1 \in \mathbf{X}(x')$ but $m_2 \notin \mathbf{X}(x')$. So *M* is totally disconnected.

Example

[See example 5.7.30] In $\mathbf{2}^4$ we have

$\mathbf{X}(\{1\}) = \{\mathbf{X}1\} = \{m_1\}$

$\mathbf{X}(\{2\}) = \{\mathbf{X}1\} = \{m_2\}$

$m_1 = \{\{2,3,4\},\{2,3\},\{2,4\},\{3,4\},\{2\},\{3\},\{4\},\varnothing\}$

$m_2 = \{\{1,3,4\},\{1,3\},\{1,4\},\{3,4\},\{1\},\{3\},\{4\},\varnothing\}$

$m_1 - m_2 = \{\{2,3,4\},\{2,3\},\{2,4\},\{2\}\}$

Let $x = \{2\}$.

$\mathbf{X}\left(\{2\}'\right) = \mathbf{X}(\{1,3,4\}) = \{\mathbf{X}1, \mathbf{X}3, \mathbf{X}4\} = \{m_1, m_3, m_4\}$

Hence $m_2 \in \mathbf{X}(\{2\}) = \{m_2\}$ but $m_2 \notin \mathbf{X}\left(\{2\}'\right) = \{m_1, m_3, m_4\}$.

2. A space *S* is compact if every open cover of *S* has a finite subcover. Let *M* be covered by a collection of open sets $\{O_\alpha\}$ where $\alpha \in A$. To aim for a contradiction, suppose *M* is not covered by some finite subset of $\{O_\alpha\}$. Each O_α is a collection of sets of the form $\mathbf{X}(x)$; hence *M* can be covered by a collection *U* of sets of the form $\mathbf{X}(x)$ where $x \in C \subseteq B$. Then no finite subset of *U* covers *M*. Let $x_1, \ldots, x_k \in C$ be any finite subset of *C*. Then, because there is no finite subcover of *M* we have: -

$\mathbf{X}(x_1) \vee \mathbf{X}(x_2) \vee \ldots \vee \mathbf{X}(x_k) = \mathbf{X}(x_1 \vee x_2 \vee \ldots \vee x_k) \neq M$

Thus $x_1 \vee x_2 \vee \ldots \vee x_k \neq \mathbf{1}$ for any $x_1, \ldots, x_k \in C$. Then the ideal generated by *C* is a proper ideal. Hence, by the Boolean Prime Ideal theorem [7.25 below] there is a maximal ideal m_C containing *C*. But this applies to any $x_1, \ldots, x_k \in C$, so $m_C \notin \mathbf{X}(x)$. Hence *M* is not covered by the collection *U* of open sets, contradicting our supposition that it was.

The Cantor set is $\mathbf{2}^\omega \cong \mathbf{2}^{\mathbb{N}_\infty}$. The topology above (U,T) for $\mathbb{N}_\infty$ is also a basis of the Cantor set, and hence shows that the Cantor set is compact. [See 5 / 3.7] We take the ideal $F(\mathbb{N})$ of all finite subsets of $\mathbb{N}$. Then we make a copy of this and add to each member the "point at infinity" ∞, which creates a collection homeomorphic to all cofinite subsets – $C(\mathbb{N})$. Together these two collections cover $\mathbb{N}_\infty$. Both collections are independently locally compact. A single set $U_\infty \in C(\mathbb{N})$ suffices to cover ∞ but by the definition of the topology it must contain some finite subset of $\mathbb{N}$ as well; all the other points may be covered by a collection of members of

$F(\mathbb{N})$. Since this collection is locally compact there is a finite subcover of it. Let this finite subcover be denoted $U_{\mathbb{N}}$; then $U_{\mathbb{N}} \cup U_{\infty}$ is a finite cover of $\mathbb{N}_{\infty}$. The set $\mathbb{N}_{\infty}$ acts as the skeleton of the Cantor set; hence the Cantor set inherits its compactness from $\mathbb{N}_{\infty}$. One-point compactification is an analogue of the Heine-Borel theorem – in its turn equivalent to the Completeness Axiom. The compactness of the Cantor set also follows from Tychanoff's theorem. (For further comment see Givant and Halmos [2009] p. 305.)

1.3 Tychonoff's theorem

Every product of compact spaces is compact. Conversely, if a product of non-empty spaces is compact then each of its factors is compact.

1.4 A puzzel and its resolution

Notwithstanding the remarks already made above, the compactness property for the Stone space has an air of paradox that should be investigated further. For finite Boolean lattices compactness follows automatically, but in the infinite case there are prima facie reasons why the Stone space, here $M = S_B$, *should not be compact*, that the theorem refutes. To explain: the Stone space, *M*, is infinitely partitioned into atoms $m_1, m_2, \ldots$ which are separated from each other, disjoint and taken together the space is totally disconnected, as shown in the first part of the theorem. Each atom is apparently related to some element $x \in B$ so that we have $m \in \mathbf{X}(x)$; also *B* is infinite. The definition of compactness requires that *every* open cover of a compact space has a finite subcover. To show that *X* is *not compact* one must find an example of a cover for *X* that is not finite; so a space is *not compact* if there exists just one infinite open cover for it, which makes the meaning clear.

<u>Example</u>

The open interval $(0,1)$ is not compact in $\mathbb{R}$. For each member of the collection $\left(\frac{1}{n},1\right)$ where $n \geq 2$ is open in $(0,1)$; also $(0,1) = \bigcup_{n=2}^{\infty}\left(\frac{1}{n},1\right)$. But there is no finite subcollection of this collection that covers $(0,1)$.

Following the example, one is inclined to conclude: *surely the collection* $\mathbf{X}(x)$ is an infinite open cover for $M = S_B$? Examining the proof of the theorem closely we see that the crucial step when we are lead out of this conclusion occurs when the Boolean Prime Ideal theorem is cited to establish the existence of a maximal ideal m_{μ} for any subset of *B*. This theorem in turn rests upon the Axiom of Choice (specifically in the form of Zorn's Lemma). The Axiom of Choice functions as a species of completion axiom; it is this principle that embeds the possibly non-atomic lattice *B* into a complete atomic lattice. Specifically, in the case of the Cantor set, $X = \mathbf{2}^{\omega}$, the ideal $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ of all finite subsets of is *extended* to a maximal ideal M, which becomes at atom of the Stone space and corresponds to a maximal element μ. (Proven below 2.1 and see also 5.7.41/42) The Stone space of the Cantor set is also isomorphic to the Cantor set again, $S_X \cong \mathbf{2}^{\omega}$. Since the two sets are isomorphic, though inverted, there must be an element in $X = \mathbf{2}^{\omega}$ corresponding to this atom $\mathrm{M} \in S_X$, which is an atom in *X.* Since S_X is an inverted copy of *X*, this must be $\lambda = \mu'$. Suppose now we wish to make an infinite open cover for *X.* Then that must include the ideal M; also λ is included in every infinite open cover for *X.* But λ is the infimum in $\mathbf{2}^{\omega}$ of the set of all cofinite subsets of ω which includes *every infinite subset of* ω. Hence, when λ is subtracted from the cover, there remains only finite subsets of ω, which because it contains only finite subsets must be compact – i.e. have a finite subcover. Thus, it is the addition of λ to $X = \mathbf{2}^{\omega}$ and the correspondent addition of M to $S_X \cong \mathbf{2}^{\omega}$ that makes both compact. Since $\lambda = \mu'$ both elements belong to both any complete infinite Boolean algebra *B* and its Stone space S_B has a corresponding maximal filter Λ and a maximal ideal M where $\Lambda = \mathrm{M}'$.

This is an exact analogy with the Heine-Borel theorem which renders the interval $[0,1]$ compact by adding the neighbourhood of 1 to the locally compact subset $[0,1)$ and thus acts as a one-point compactification of it. In the Cantor set, $X = \mathbf{2}^{\omega}$, μ represents the neighbourhood of **0** and λ the neighbourhood of **1**. The addition of λ to $\mathbf{2}^{<\omega}$ completes it by embedding $\mathbf{2}^{<\omega}$ in $\mathbf{2}^{\omega}$ and allowing a complete set of atoms for $\mathbf{2}^{\omega}$.

1.5 Result

The clopen subsets of a Boolean space *X* form a field of sets.

1.6 Definition, dual Boolean algebra

The field of clopen subsets of a Boolean space *X* is called the *dual (Boolean) algebra,* denoted B_X.

1.7 Result, Stone duality

The dual algebra B_X of the field of clopen subsets of S_B is isomorphic to the original algebra *X.* (Proof, Mendelson [1970] p.171) Additionally, *X* and S_B are homeomorphic. (Proof, Mendelson [1970] p.171)

I remark on the subtle difference between this theorem and the preceding Theorem 1.2 demonstrating the existence of the Stone space. Here we start with a Boolean space *X* and construct its dual algebra B_X. We then construct the Stone space, S_B, of this dual and the claim is that $X \cong S_B$. In the preceding result we start with a Boolean space *B* and construct its Stone space. A relation of isomorphism does not necessarily exist between *B* and this Stone space S_B. This latter case is illustrated by the relation between the Boolean algebra, $FC(\omega)$, of all finite and cofinite subsets of ω and the Cantor set, $\mathbf{2}^{\omega}$. The two are not isomorphic, but the Cantor set is the Stone space of $FC(\omega)$.

2 Partition of the Cantor set

Let M denote the maximal ideal of all finite subsets of $\mathbb{N}$; it cannot be a principal ideal in the sense that it cannot have a principal element $\mu \in \mathbb{N}$.

2.1 Proof of this assertion

To show this, note that the maximal ideal, M, consists of all subsets of $\mathbb{N}$ that do not contain some fixed element $b \in \mathbb{N}$. But every singleton $\{b\} \in \mathrm{M}$ (where J is the ideal of all finite subsets of $\mathbb{N}$ as above). If M has a principal element u then the meet of any two elements must lie in M. That is for $x, y \in \mathrm{M}$ we have $x \vee y \le u$ where $\mathrm{M} = J_{\mu}$. Now $\{b\} \in \mathrm{M}$ and $\mathbb{N} - \{b\} \in \mathrm{M}$ hence $\mathbb{N} = \{b\} \cup \{\mathbb{N} - \{b\}\} \in \mathrm{M}$.

The one point compactification of $\mathbb{N}$ is: -

$\mathbb{N}_{\infty} = \mathbb{N} \cup \{\infty\} = \{0,1,2, \ldots, \infty\}$

At this point I will introduce the result that we may identify the "point at infinity" $\infty = \mathbb{N}$, which shall be justified later [See 2.4 and 2.5 below]; assuming this result, the one-point compactification may be written: -

$\mathbb{N}_{\infty} = \mathbb{N} \cup \{\mathbb{N}\} = \{0,1,2, \ldots, \mathbb{N}\}$

μ and $\mathbb{N}$ are different descriptions of the same underlying potentially infinite collection of all natural numbers, so their identification is possible if we "forget" the difference of order relations upon them. Hence we may also write: -

$\mathbb{N}_{\infty} = \mu \cup \{\mu\} = \{0,1,2, \ldots, \mu\}$

and since we have defined $\lambda = \mu' = \mathbb{N}_{\infty} - \mu = \{\mu\}$, this also gives: -

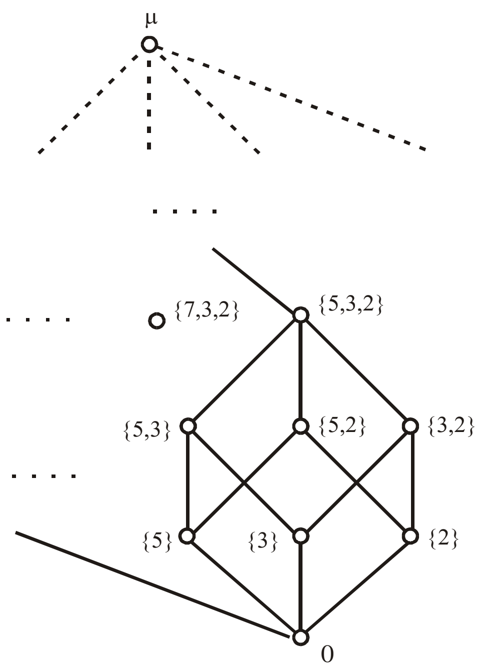
$\mathbb{N}_{\infty} = \mu \cup \lambda$

We are therefore in a position to show that there is an element $\mu \in \mathbb{N}_{\infty}, \mu \notin \mathbb{N}$ which is both principal and maximal. Every set is closed in itself, hence $\mathbb{N}$ is a closed subset of itself. M is the closure of the set of all finite subsets of M, so we are in a position to define $\mathrm{M} = \mathbf{P}(\mathbb{N})$. The discussion above identified μ as the unordered collection of members of $\mathbb{N}$; so under the "forgetful" identification $\mu = \mathbb{N}$ we also have $\mathrm{M} = \mathbf{P}(\mu)$. Then, by the Axiom of Foundation,

$\mathbb{N} \notin \mathbb{N}$ $(\mu \notin \mu)$, yet $\mathbb{N} \in \mathbb{N}_\infty$ $(\mu \in \mathbb{N}_\infty)$. Repeating the above proof but this time with M as the maximal ideal of all finite subsets of $\omega \cong \mathbb{N}_\infty$ we have $\lambda = \{\mu\} \notin \mathrm{M}$ as the distinctive atom of the Cantor set that is not in M so the above proof does not go through.

This is what we would expect. The ideal M is an atom of the Stone space $S(F(\mathbb{N}))$ which is an inverted isomorphic copy of the Cantor set, $\mathbf{2}^\omega$; so there must be an atom in the Cantor set to which this ideal M corresponds. Putting $\mathrm{M} = \mathbf{P}(\mu)$ and observing $\lambda = \mu'$ we see that λ is the atom in $\mathbf{2}^\omega$ that is the inverted corresponding image of M in the Stone space, $S(F(\mathbb{N}))$. This permits the following diagrammatic representation of M : -

Maximal ideal of all finite sets

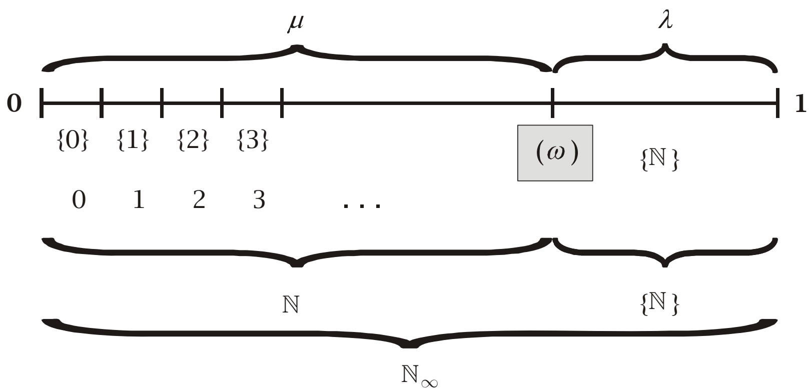


The relation between $\mathrm{M} = \mathbf{P}(\mu) = J_\mu$ and λ reflects the usual relationship in a lattice, as indicated by the following result: A principal ideal J_p is maximal if and only if p' is an atom. (Givant and Halmos [2009] p. 172)

2.2 The "meaning" of the Axiom of Choice

The Axiom of Choice is a species of completion axiom required for actually infinite collections. The partition of $[0,1]$ into ω segments creates a unit of measure (metric – see

[Chap. 4 /5.23 and Chap 2. /2.9.8]); disjoint adjacent atoms of the partition lie a distance of 1 unit apart under this measure. Relative to this measure it is not possible to distinguish the boundary between atoms that are a finite distance apart and atoms that are ω units apart. There is an *indeterminate boundary between the finite and the infinite partitions of* ω that cannot be measured. Let us mark this indeterminate boundary by (ω). Then the Axiom of Choice (or equivalent) asserts that (ω) may be regarded as a partition of determinate sets.



2.3 Partition of the Cantor set

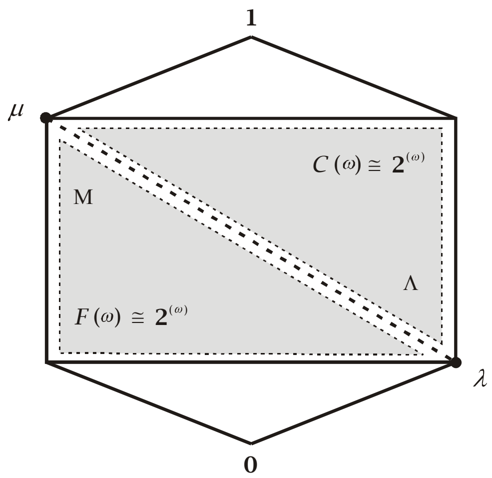
We may apply the principle of Stone duality [1.7 above] also to infer the existence of an ultrafilter with minimal element $\lambda = \mu'$. This ultrafilter contains all co-finite subsets of ω. Another term for ultrafilter is *prime filter*. Thus Λ is a prime filter and M is a prime ideal. Neither μ nor λ are actual subsets of $F(\mathbb{N})$ and $C(\mathbb{N})$ respectively; they both stand for limits; in a sense μ represents the boundary between the finite and infinite subsets of ω and λ represents the boundary between the co-finite and the not-co-finite subsets of ω. The Cantor set $\mathbf{2}^\omega$ is partitioned into three "regions": -

1. $F(\mathbb{N}) \cong \mathbf{2}^{<\omega} \subset \mathrm{M}$, the prime ideal of all finite subsets of $\mathbb{N}_\infty$.
2. $C(\omega) \cong \mathbf{2}^{<\omega} \subset \Lambda$, the prime filter of all co-finite subsets of $\mathbb{N}_\infty$.

   Let $FC(\omega)$ denote the Boolean sub-algebra of all finite and co-finite subsets of $\mathbb{N}_\infty$; we have $FC(\omega) \equiv F(\mathbb{N}) \cup C(\omega)$.
3. $\mathbf{2}^\omega - FC(\mathbb{N})$, the partition of $\mathbf{2}^\omega$ comprising all infinite subsets of ω that are not co-finite. I shall call this the *boundary set* of the Cantor set.

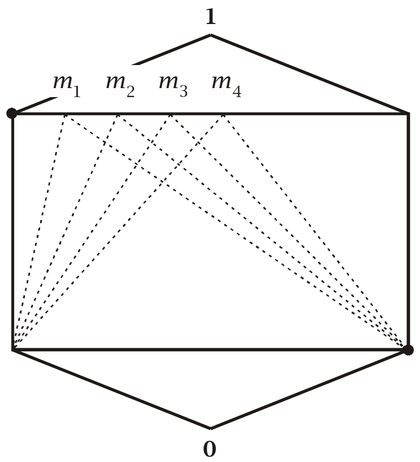
I use the symbol $F(\mathbb{N})$ to denote the finite subsets of $\mathbb{N}_\infty$; this is because they can be defined on $\mathbb{N}$ alone, without regard to its embedding in $\mathbb{N}_\infty$. However, we have (given the Axiom of

Choice) $F(\mathbb{N}) \cong F(\mathbb{N}_\infty) \cong F(\omega)$ so the three notations are interchangeable. The cofinite sets cannot be defined without regard to $\mathbb{N}_\infty \cong \omega$, because they are the sets $\mathbb{N}_\infty - V$ where $V \subseteq \mathbb{N}$. (Alternatively, $\omega - V$.) Hence, we write this set as $C(\mathbb{N}_\infty) \cong C(\omega)$. Given the Axiom of Choice the two symbols are interchangeable, and unless we have the Axiom of Choice the notion of the filter $C(\mathbb{N}_\infty)$ makes no sense at all because we cannot then have the one-point compactification of $\mathbb{N}$. [See 5.3.12]

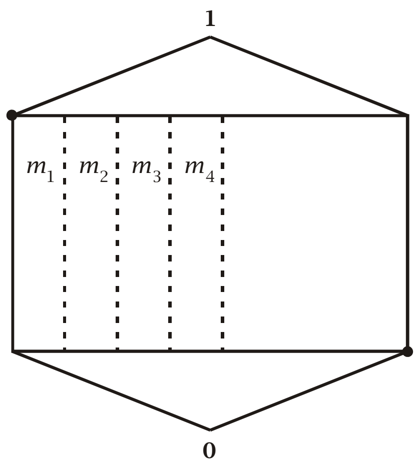


The ideal $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$ and filter $C(\omega) \cong \mathbf{2}^{<\omega}$ are non-atomic generalised Boolean algebras. Here non-atomic is to a discrete structure as open is to a continuous one (manifold). Both may be completed: this means, to extend and embed $F(\mathbb{N})$ in its maximal ideal (prime ideal) M and likewise to embed $C(\omega)$ in its maximal filter (ultrafilter) Λ. Together $\{\mathrm{M}, \Lambda\}$ partition the Cantor set. The boundary of the Cantor set is shown in the above diagram as an "open" line joining the maximal element (co-atom) μ of M to the atom λ of Λ. The diagram has been draw to emphasise the metric relation in the lattice: in the metric the distance of μ from **1** is one unit, and likewise the distance of λ from **0** is one. The boundary is a huge set in comparison to $FC(\omega) \equiv F(\mathbb{N}) \cup C(\omega)$, which is an atomless, countably infinite Boolean algebra (with **1**). Thus the boundary, $\mathbf{2}^{\omega} - FC(\mathbb{N})$, is a set of the same cardinality as the Cantor set – in other words, of cardinality $\mathbf{2}^{\aleph_0}$. The boundary contains all the infinite sets that are not co-finite. For example, the algebra of the Countable collection of prime generated infinite and coinfinite sets, $PI(\omega)$, [See 5 / 5.11] is contained in the boundary.

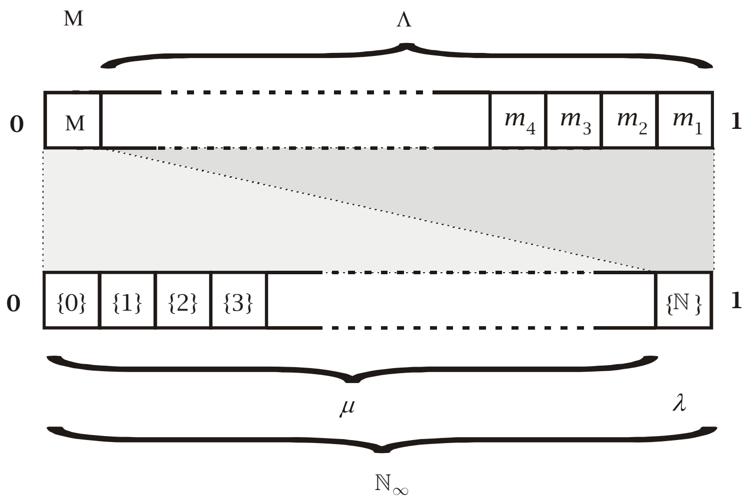
We are now in a position to clarify the relations of duality in regard to the Stone representation theorem. Conceptually we begin with an attempt to partition the interval $[0,1]$ by the potentially infinite collection, $\mathbb{N}$. This creates a locally compact, totally disconnected partition, so it is not quite a Boolean space (which must be compact) but *almost so*. Ignoring this subtle difference for the moment we obtain a generalised Boolean algebra defined upon it as $F(\mathbb{N}) \cong \mathbf{2}^{<\omega}$. This is non-atomic. If in fact we expand this to the algebra $FC(\omega) \equiv F(\mathbb{N}) \cup C(\omega)$ we have implicitly already completed the algebra as a whole and added the boundary, though, in a sense, *we do not know this yet.* It is the Stone representation theorem that makes this apparent. In a Boolean lattice ideals "expand" downwards, as in the following diagram: -



But for heuristic purposes we may think of the maximal ideals as vertical partitions of the Boolean lattice, thus: -



The partition constitutes the Stone space, which also acts as the completion of the original Boolean space. The complete Boolean algebra constructed as the field of sets over the original Boolean space is dual to the complete Boolean algebra constructed over the Stone space of maximal ideals.



2.4 (+) The size of λ

We have started to identify $\lambda = \{\mathbb{N}\}$, where $\mu = \mathbb{N}$ under the Axiom of Choice. That λ must be an infinite collection is implied by the Stone duality, and illustrated by the above diagram; however, the size of λ is possibly in question, owing to the following remarks, which are derived from Givant and Halmos [2009]. They are discussing *canonical extensions*: -

2.5 Size of a canonical extension

"Suppose A has an infinite number m of elements. It can be shown that the number of ultrafilters in A is between m and 2^m. Each ultrafilter determines, and is determined by, a unique atom in the canonical extension, and each element in the canonical extension determines, and is determined by, a unique set of atoms. There are therefore as many elements in the canonical extension as there are subsets of the set of ultrafilters in A. Conclusion: the canonical extension has between 2^m and 2^{2^m} elements." (Givant and Halmos [2009] p. 197.)[1]

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[1] Summary relating to canonical extensions (Givant and Halmos [2009] p. 197.)

(1) Result: Let X be the set of 2-valued homomorphisms of any Boolean algebra A. Then A can be embedded in $\mathbf{P}(X)$. That makes $\mathbf{P}(X)$ an extension of A. Let the embedding be $f : A \to \mathbf{P}(X)$.

Properties: -

1. $\mathbf{P}(X)$ is atomic and complete.
2. Every element $p \in A$ has an image $f(p) \in \mathbf{P}(X)$.
3. Any two distinct atoms $q, r \in \mathbf{P}(X)$ are separated by some element $p \in A$, meaning $q \le p, r \le p'$. Furthermore, the atoms have the form $q = \{x\}, r = \{y\}$ for distinct 2-valued homomorphisms $x, y \in X$.
4. $\mathbf{P}(X)$ is compact with respect to A.

Let $E \subseteq A$. Suppose $\sup(E) = q \in A$, then there exists a finite subset F of E such that $\sup(F) = q \in A$.

The remark places the size of λ as between $\aleph_0$ and $2^{\aleph_0}$, so it might be $\aleph_1 \neq 2^{\aleph_0}$ for aught we know. However, we can see immediately that this is not possible; λ is the infimum of all co-finite sets in ω; it is the *last* element of the chain $\omega - \{0\}, \omega - \{0,1\}, \omega - \{0,1,2\}, \ldots$; we cannot subtract a set of cardinality $\aleph_0$ from ω to obtain a set of cardinality $\aleph_1$ - this would violate the Pigeon-hole principle [See Chap.15 /3.1]. So λ is a set of cardinality $\mathbb{N}$. It remains to show that, relative to μ, it is already an ordered set. However, λ only exists if we grant the Axiom of Choice, which is then available to order λ. Hence $\lambda \cong \mathbb{N}$. We can go further and give a complete *intensional* characterisation of λ. The Stone duality shows that λ is equivalent to a collection of atoms for the ideal in the dual space of maximal ideals given by $S_M - \mathrm{M}$ where M is the maximal ideal of all finite sets. A basis for this ideal is the set of all co-atoms of the ordinal algebra: -

$\lambda = \{\omega - \{0\}, \omega - \{1\}, \omega - \{2\}, \ldots\} \cong \mathbb{N}$.

2.6 The sub-algebra of finite and co-finite sets

[Subalgebras were defined at 5.7.13.] The theory of ordinals is based on the notion of order-type, itself inferred from the properties of well-ordered sets. According to this theory it is possible to count up to ω but not to count down from it. For example, $1 + \omega \sim \omega$, where $\sim$ denotes the *similarity relation* [Chap. 2 / 2.8.2] whereas $\omega + 1 > \omega$. Counting down from ω is undefined and the expression $\omega - 1$ is meaningless. Nonetheless, in the theory of ordinals it is customary to define the successor ordinal by means of the set union operation: $x' = x \cup \{x\}$. From this we obtain the sequence of ordinals. This definition gives a derivative sense to the idea of counting down from ω - we can form a subset of ω by taking the difference between ω and any finite set; this generates the collection of *cofinite subsets of* ω. Some examples of cofinite sets are: -

$\omega - \{0\} \equiv \{1,2,3,4,5,\ldots\}$ $\omega - \{1\} = \{0,2,3,4,5,6\ldots\}$ $\omega - \{1,2,5\} = \{0,3,4,6,7,\ldots\}$

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(2) Definition, atom separation property: An extension of a Boolean algebra *A* is said to have the *atom separation property* with respect to *A* if any two atoms *q* and *r* in the extension are separated by some element *p* in *A* in the sense $q \le p, r \le p'$. (3) Definition, compactness property: An extension is said to have the *compactness property* with respect to *A* if whenever a subset *E* of *A* has a supremum in the extension, say *q*, that belongs to *A*, then some finite subset of *E* has the same supremum. (4) Definition, canonical extension: A complete and atomic Boolean extension of an algebra *A* that has the atom separation and compactness properties is said to be a *canonical extension* or a *perfect extension* of *A*. (5) Theorem: Every Boolean algebra has a canonical extension. (6) Remark: Every Boolean algebra has a complete and atomic extension. For any given Boolean algebra there is just one canonical extension up to isomorphism. (7) Lemma: If *B* is a canonical extension of a Boolean algebra *A*, then the distinct atoms in *B* are precisely the infima of the distinct ultrafilters in *A*. (8) Remark: The set of atoms in *B* is in bijective correspondence with the set of ultrafilters in *A*. (9) Theorem: Any two canonical extensions of a Boolean algebra *A* are isomorphic via a mapping that is the identity on *A*. (10) Remark: A finite Boolean algebra is its own canonical extension and there is no increase in size.

Such sets can be recursively enumerated; evidently, they are all infinite collections. Hence, at one "end" of the lattice $\mathbf{2}^{\omega} \cong \{0,1\}^{\omega}$ we have the finite sets, which form the collection of all finite sets, and at the other end we have the collection of all cofinite sets. Unions of finite sets correspond to intersections of co-finite sets; for example: -

$\{1\} \cup \{2\} = \{1,2\}$ $\{1\}' \cap \{2\}' = \{1,2\}' \equiv \{0,2,3,4,...\} \cap \{0,1,3,4,...\} = \{0,3,4,5...\}$

In addition to the regions of finite and co-finite sets, *there is also a region between both of these* that comprises sets that are infinite but not cofinite. I call this region *the boundary.* When we descend from infinity by deleting finite numbers of elements we never reach a set of non-infinite cardinality. When we ascend from a finite set by adding a finite number of elements we never reach beyond a set of finite cardinality. The two processes cannot meet in the middle, so the structure is incomplete.

2.7 Example

Let $FC(\omega)$ be the field of sets of all finite and cofinite sets of positive integers. $FC(\omega)$ is a subfield of the field of sets $\mathbf{P}(\omega) \cong \mathbf{2}^{\omega}$. Let *C* be the set of all singletons of the form $\{n\}$ where *n* is a positive integer.

1. The join of *C* in $P(\mathbb{N})$ is $\omega - \{0\} = \{1,2,3,...\}$.
2. The join of *C* in $FC(\omega)$ is ω since $\omega - \{0\} \notin FC(\omega)$.

In this case there does exist a supremum for the subset *C* in *B*, but there is another element not contained in *B* that could also act as this supremum if *B* were complete. Here we see that the join of a set in the field of sets is not necessarily the union. The join of a set in one Boolean algebra is not necessarily the same as the join in another. Strictly we should show that join is relative to the given algebra *B* and write $\bigvee_{x \in A}^{B} x$. In this case the join of *C* does exist in $B = FC(\omega)$ because the set $\mathbb{N}$ is a unique join (joins must be unique).

2.8 Result, interval algebra

Let *A* be the interval algebra of the real numbers, and let *E* be the subset of *A* consisting of all intervals $[n, n+1)$, where *n* ranges over the integers. The subalgebra *A* generated by *E* is isomorphic to the field of finite and cofinite sets of integers. (Givant and Halmos [2009] p. 102)

2.9 The countable collection of prime generated infinite and cofinite sets

Countable collection of prime generated infinite and coinfinite sets , $PI(\omega)$, was introduced at 5.5.11. It comprises only sets that belong to the boundary; one such is the actually infinite complete set of all even numbers: -

$[2] = \{2,4,6,8,...\} = \{x \in \mathbb{N} : x = 2n \text{ for some } n \in \mathbb{N}\}$

This is not cofinite because we could never reach this set by either (*a*) adding a finite number of elements to a finite set, or (*b*) removing from ω a finite number of elements; so it must always differ from any finite or cofinite set by an infinite number of members – this being an aspect of the property of infinity known as its *inexhaustibility*. Other infinite yet not cofinite sets include: -

$[1] = \{1,2,3,...\}$ $\quad [6] = [2] \cap [3] = \{0,6,12, ...\}$ $\quad [30] = [6] \cap [5] = \{0,30,60, ...\}$

$y_0 \equiv [2] - [6] = \{2,4,8,10, ...\}$ $\quad y_1 \equiv [30] - [6] = \{6,12,18,24, ...\}$

Here $y_0 \wedge y_1 = y_0 \cap y_1 = \varnothing$ and $[1] \cup [2] = [1] \vee [2] = \mathbf{1} \cong \omega$. This illustrates the result that the set of all infinite but not cofinite subsets of ω is a Boolean algebra in its own right. Although $[6]$ lies in the filter generated by $[2]$ no addition of any finite number of elements to $[2]$ will suffice to construct $[6]$; we can in fact only reach one infinite (not cofinite) subset of ω from another by an infinite "limiting" process. So each infinite (not cofinite) subset of ω is itself an isomorphic copy of the set of all finite subsets of ω.

2.10 Example

Let $FC(\omega)$ be the Boolean algebra of all finite and cofinite subsets of ω. Let (2) be the set of all sets of the form $\{2n\}$ where $n \in \mathbb{N}$. Then there is no join of (2) in $FC(\omega)$.

<u>Proof</u>

Suppose that there is a join $u \in FC(\omega)$. Then $u = \bigvee_{x \in (2)} x$, and u would have to contain all positive even integers. Since u is also co-finite it must all but finitely many positive integers. Then any proper subset of u that is obtained by removing an odd integer would also be an upper bound for (2). This contradicts the assumption that u is the least upper bound of (2).

The join must be an infinite but not co-finite subset of ω. It contains nothing but even numbers and is actually infinite: $[2] \equiv \{2,4,6,...\}$. This illustrates that $FC(\omega)$ is not complete.

3 The Cantor set

The Cantor set occupies a unique mid-position between our two conceptions of infinite analytic logic. On the one hand it is the maximal structure of the analytic logic of ω and *is the Boolean lattice* over which that logic is built; on the other hand it is a necessary subspace of the continuum and therefore a fundamental sub-structure over which the analytic logic of the continuum is constructed.

3.1 (+) Aside, the analytic logic of the continuum

There appears to be an analytic logic of the continuum that is not expressed, for instance, in the formal analytic logic of the predicate calculus. Analytic logic is based on the fundamental idea of containment. The predicate logic interprets this principle by means of an analysis of the continuum into a discrete skeleton of ω parts over which a lattice is constructed and on top of that analytic logic is built. But there are analytic relations in the continuum not encompassed or seen by this logic.

3.2 Iterative definition of the Cantor set

Because of this construction the Cantor set is also known as the *Cantor ternary set*, and also designated $SVC(3)$, denoting a Smith-Voltarra-Cantor set.[2] Let $F_1 = [0,1]$. Remove from this the middle open third, that is the interval $\left(\frac{1}{3},\frac{2}{3}\right)$, to obtain $F_2 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$. Remove from each part of this its middle open third, to obtain $F_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9} \cup 1\right]$. Iterate this process to obtain a sequence of closed sets F_n each of which contains all of its successors. Define the *Cantor set* by $\mathbf{2}^{\omega} = \bigcap_{n=1}^{\infty} F_n$. This is a closed set. F contains all the points that remain after all the open intervals in the sequence $\left(\frac{1}{3},\frac{2}{3}\right), \left(\frac{1}{9},\frac{2}{9}\right), \left(\frac{7}{9},\frac{8}{9}\right), \ldots$ have been removed. It therefore contains all the end-points of these intervals, $0, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \ldots$. It can be shown that the Cantor set "contains a multitude of points other than the above end-points, for the set of these end-points is clearly countable, while the cardinal number of F itself is c, the cardinal number of the continuum." (Simmons [1963] p. 67.)

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[2] Smith-Volterra-Cantor sets: These are examples of perfect, nowhere dense sets. The first construction was by Smith in 1875, the second by Volterra in 1881, but they only became widely known after Cantor rediscovered the construction in 1883. They will be abbreviated as SVC sets. (1) The general construction of SVC sets: The Cantor set is an example of a Smith-Voterra-Cantor set, denoted *SVC* sets. SVC sets are constructed by starting with a closed interval and then removing an open subinterval; then from the remaining subintervals another open subinterval is removed; the process is iterated indefinitely. The SVC set is the intersection of the countably infinite collection of sets that remain after each iteration. (2) The Cantor set as an *SVC* set: Cantor's ternary set belongs to a family of SVC sets such that at the *k*th iteration, an open interval of length $\frac{1}{n^k}$ is removed from the centre of each remaining closed interval. The resulting set may be denoted $SVC(n)$ where $n \geq 3$. Cantor's ternary set is $SVC(3)$. In the ternary construction for $SVC(3)$ if we replace the 3 in this expression by a 4 we obtain $SVC(4)$.

3.3 The puzzle of the Cantor set

This remark by Simmons is indicative of a significant puzzle when we consider the Cantor set. The members of this set are all enclosed within intervals of zero measure, whose endpoints are a set of cardinality $\aleph_0$; nonetheless the cardinality of the members of the Cantor set is $2^{\aleph_0}$. How is it possible for $2^{\aleph_0}$ points to be enclosed in $\aleph_0$ intervals?

3.4 Result, ternary representation of an element of the Cantor set

The elements of the Cantor ternary set are those numbers that can be written in base 3 without using the digit 1.

<u>Proof</u>

Consider the base 3 expansion of numbers between 0 and 1. This expansion uses only the digits 0, 1 and 2.

<u>Example</u>

$$0.21_3 = \frac{2}{3} + \frac{1}{3^2} = \frac{7}{9}$$

Call such a base 3 representation of a fraction a *ternary fraction*. Observe also that any finite ternary fraction can be written as infinite ternary fraction with repeating 2s. For example, the ternary fractions

$0.1_3 = 0.0222..._3$ $\qquad$ $0.01_3 = 0.00222..._3$

Now consider the construction of the Cantor set.

Iterate	Remove	Removed as an infinite ternary fraction*	What is left as an infinite ternary fraction
1	$\left(\frac{1}{3}, \frac{2}{3}\right)$	$(0.1000...)$ to $0.111...$	0.000... to 0.0222... 0.2000... to 0.222...
2	$\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$	$(0.01000...)$ to $0.01111...$ $(0.21000...)$ to $0.2111...$	0.000... to 0.00222... 0.02000... to 0.0222... 0.2000... to 0.20222... 0.22000... to 0.2222...

<u>Note</u>

* 0.1 is *not removed*. That is why I have put it in brackets. What is removed is the open interval from 0.1 to 0.111... etc. We see that at every iteration the ternary fractions that we remove are those that as infinite (*but not finite*) ternary fractions contain a 1, and those that are left are those that as finite or infinite ternary fractions use only the digits 0 and 2. After a countably infinite number of iterations we have as the elements of the Cantor set only those numbers that can be written in base 3 *as infinite ternary fractions* without using the digit 1.

3.5 Example

The fraction $\frac{1}{4}$ is contained in the Cantor set and yet is not an end-point of any closed interval in it. To show this, observe that: -

$$0.0202\overline{02}_3 = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^9} + \ldots = \frac{2}{9} \cdot \frac{1}{1 - 1/9} = \frac{2}{9} \cdot \frac{9}{8} = \frac{1}{4}.$$ [3]

By examination of the iterative process above we see that the end-points of any closed interval in the Cantor set terminate in strings of the form: -

0000... or 2000.... or 2222

The first case only occurs in the case of $0 = 0.000\ldots$. Thus any infinite ternary fraction that can be written with 0 and 2s that recur in any other pattern is an element contained in the Cantor set, but *not* identical to an end point of any closed interval in the Cantor set. This example illustrates *the puzzle of the Cantor set.* [3.3 above]

3.6 Result, canonical representation of the Cantor set, the Devil's staircase

Let *x* be an element in the Cantor set. Take the base 3 expansion of *x*. Convert each 2 in that expansion to a 1 and then read the number as a base 2 number. This function is said to be the *Devil's Staircase*, also called the *Lebesgue singular function* and maps all of *C* onto $[0,1]$.

<u>Example</u>

$$\left(\frac{2}{3}\right) = 0.2_3 \to 0.1_2 = \left(\frac{1}{2}\right)$$

Thus the *canonical representation of the Cantor set* is the set $\mathbf{2}^\omega = \{0,1\}^\omega$. The set $[0,1)$ is equinumerous with the Cantor set. There are *c* many points in the Cantor set.

<u>Proof</u>

The Cantor construction starts with an interval $I \cong [0,1]$ with end points $x \in \{0,1\}$. At the first iteration this is subdivided into closed and open sets with end points:

$$\frac{1}{3}\{0,1,2,3\} = \frac{1}{3}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

At the second iteration this is further subdivided into closed and open intervals with end points: -

[3] This uses a geometric series

$$\frac{1}{9}\{0,1,2,3,4,5,6,7,8\} = \frac{1}{9}\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

$$= \frac{1}{3^2}\left\{a_1 2^0 + a_2 2^1 + a_3 2^2\right\} \text{ where } (a) = (a_1, a_2, a_3) \text{ is a binary sequence}$$

And so on. So any such binary sequence may be mapped to a real number by: -

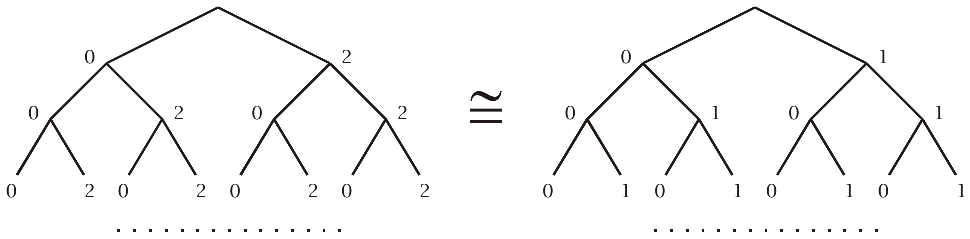
$f:(a_n)\to \frac{1}{3^N}\sum_{n=1}^{N} 2^n a_n$ and in the limit, $f_\infty : \lim_{N\to\infty}(a_n) \to \lim_{N\to\infty}\left\{\frac{1}{3^N}\sum_{n=1}^{N} 2^n a_n\right\}$.

This displays the isomorphism of the set of reals to the end-points of the Cantor set, and any *SVC* set [Defined, footnote 2 above]. However, in the Cantor set certain intervals are designated as in the set and certain intervals are designated as not in the set. When *n* is finite, each interval that is in the set is designated by two finite binary sequences. For example, the first interval at the 2nd iteration is: -

$$\left[\frac{0}{9},\frac{1}{9}\right] = \frac{1}{9}\left[\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right]$$

But in the limit as $n\to\infty$ the two adjacent end-point sequences converge on a unique single real number with a single binary representation. Each of these points is in he Cantor set. So the mapping *is* a mapping of the Cantor set onto the reals. Each interval in the Cantor set is identified with a unique real number.

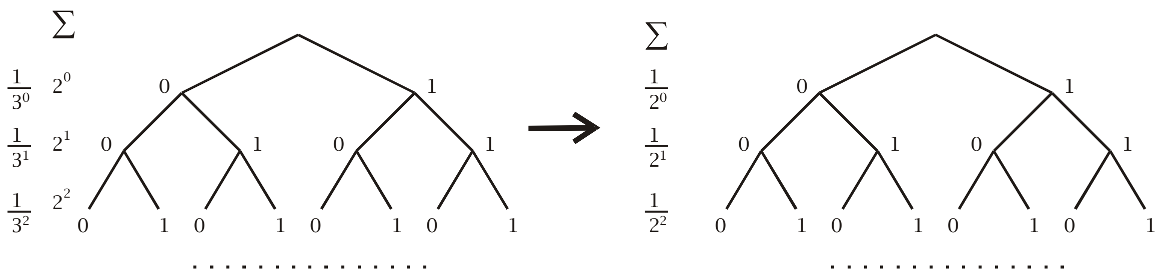
TERNARY REPRESENTATION OF THE CANTOR SET   BINARYREPRESENTATION OF A REAL NUMBER



3.7 The puzzle

The Cantor set contains continuum many, *c*, additional points that are not its end points. [See Chap. 2 / 2.7.10.] We have seen that that $\frac{1}{4}\in \mathbf{2}^\omega$ is one such point. It may be noted, that whilst the Cantor set contains *c* many points, the length of the segment removed is equal to 1, since $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots=1$. Conversely, when we remove the Cantor set from the interval $I \cong [0,1]$ we remove a set with *c* many points but leave an interval of length $1=\left|[0,1]\right|$ behind.

Diagram of the Devil's staircase



The end-points of the Cantor set are given on the left, and the elements of 2^{ω} are given on the right. It is said that there are $\aleph_0$ points in the left tree and $\left|2^{\omega}\right| = 2^{\aleph_0}$ in the right tree. This appears to be a paradox. That there are $\aleph_0$ points on the left is argued that there are countably infinite branches, each designating an end-point. That there are $2^{\aleph_0}$ points on the right is argued by the claim that it is the power set of $\mathbf{2} = \{0,1\}$.

3.8 Resolution of this paradox

The resolution is that the number and length of the branches on the left is <u>not</u> ω but $\mathbb{N}$; on the right it is ω. So there are many more points on the <u>right</u> than on the <u>left</u>. This gives an injection of the set of end-points with the set $2^{<\omega}$. The end points are generated by $\mathbb{N} \sim (<\omega)$ iterations and the points themselves are generated by ω iterations. The result of both limiting processes is the same – to produce closed sets of zero measure.

3.9 (+) Theorem, $\omega \neq \mathbb{N}$

We have just demonstrated $\omega \neq \mathbb{N}$.

<u>Proof</u>

By contradiction. Assume $\omega = \mathbb{N}$, then the number of iterations in the construction of the end-points of the Cantor set is equal to the number of iterations in the construction of the Cantor set itself. This implies $\{0,1\}^{<\omega} = \{0,1\}^{\omega}$ $\left(2^{\mathbb{N}} = 2^{\omega}\right)$, whence $\aleph_0 = 2^{\aleph_0}$.

What this illustrates is that the distinction between $\mathbb{N}$ and ω is essential to mathematics and is everywhere implicit. It shows up, for instance, in the distinction between ω and $<\omega$, the latter marking the disguised presence of $\mathbb{N}$. The Cantor set is defined by an actual, completed construction, and is the product: $\mathbf{2}^{\omega} = \{0,1\}^{\omega}$; the end-points of the Cantor set in the ternary construction are defined by a recursive (inductive) procedure over the unbounded, potentially infinite collection $\mathbb{N}$, and hence may be written $2^{<\omega}$ which makes the set of end points into the sub-direct product of an indeterminate but countably infinite number of copies of $\mathbf{2} = \{0,1\}$. Equating the two leads to paradox. This solution is also reflected in set theory in the distinction between ordinal and cardinal exponentiation

3.10 Examples of ordinal multiplication

Ordinal mulitplication is based on the concept of order types.

1. $2 \times \omega = \omega$ (ordinal multiplication).

   $2 \times \omega$ is ordered by $\langle 0,0\rangle, \langle 1,0\rangle, \langle 0,1\rangle, \langle 1,1\rangle, \ldots$ which is order-isomorphic to ω.

2, $\omega \times 2 = \omega + \omega$ (ordinal multiplication).

   $\omega \times 2$ is ordered by $\langle 0,0\rangle, \langle 1,0\rangle, \langle 2,0\rangle, \ldots ; \langle 0,0\rangle, \langle 0,1\rangle, \langle 0,2\rangle, \ldots$ which is order-isomorphic to $\omega + \omega$.

3.11 Ordinal exponentiation is not the same as cardinal exponentiation

Cardinal arithmetic is concerned with the operations of union, Cartesian product and X^Y, the class of all functions from Y into X. Therefore, ordinal exponentiation $\exp(\alpha, \beta)$, in spite of the ambiguous notation, has nothing to do with the operation of forming X^Y. Thus, for example:

Ordinal exponentiation $\exp(2, \omega) = \omega$

Cardinal exponentiation $2^\omega \succ \omega$

Regarding $\exp(2, \omega) = 2^\omega$ (ordinal exponentiation). This proceeds by induction: -

$1 \times \omega = \omega$
$n \times \omega = \omega \Rightarrow (n \times 1) \times \omega$
For all $n \in \omega, n \times \omega = \omega$
$2^\omega = \omega \times \omega = \omega$

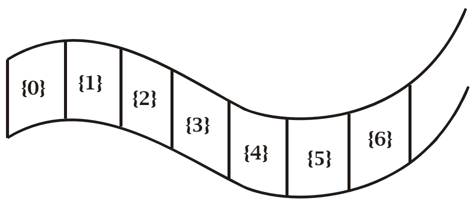
In set theory ordinal exponentiation is extended by definition to transfinite ordinals. The ordinal exponent 2^ω (note, not bold type) should perhaps be better written $2^{<\omega}$. It is a potentially infinite collection equinumerous to $\mathbb{N}$.

3.12 Cantor space

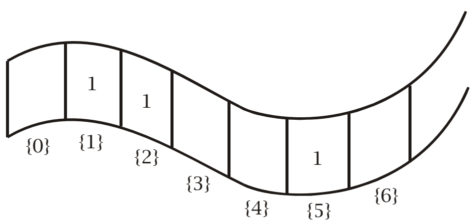
There is a good deal more that can be said about the Cantor set and the Cantor space with which it is synonymous. This material is extraneous to our purpose here – which is to examine the validity of Poincaré's thesis. I note in passing, Brower's theorem: A topological space is a Cantor space iff it is non-empty, perfect, compact, totally disconnected and metrizable. Explication of this result takes us too far from our subject, which requires only examination of the Cantor set as the essential model of formal, analytic logic, and provides us with a vehicle to demonstrate the validity of the thesis that complete induction is not a principle of analytic reasoning.

4 The model of the Cantor set

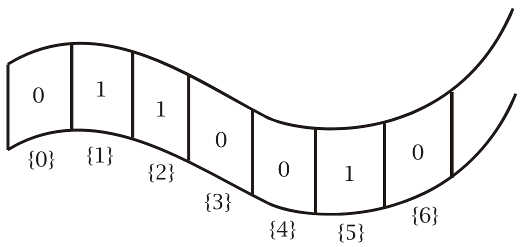
It will be useful to be able to visualise the structure of the Cantor set indicated by the preceding sections and the Stone duality in particular. Recall that our base partition is of the real line into actually $\mathbb{N}_\infty \cong \omega$ parts, each of which is an atom. This generates the Cantor space $\mathbf{2}^\omega$ as the Boolean lattice over which the analytical logic of ω partitions is constructed. Each atom corresponds to a proposition $\alpha_0, \alpha_1, \alpha_2, \ldots$ each of which, in turn, corresponds to a partition of the finite space $[0,1]$. Recall that it is an assumption of this model that there is an actually infinite partition of a finite space, $[0,1] \cong -\infty \cup \mathbb{R} \cup +\infty$ and this partition has $\mathbb{N}_\infty \cong \omega$ segments. We may represent the partitions by singleton sets: $\{0\}, \{1\}, \{2\}, \ldots$ and the correspondence gives $\alpha_0 \leftrightarrow \{0\}, \alpha_1 \leftrightarrow \{1\}, \alpha_2 \leftrightarrow \{2\}, \ldots$. Imagine a tape divided into segments, and imagine that each segment corresponds to an atom with its corresponding label.



Under the hypothesis of an infinite division of $[0,1]$ into ω parts, this is a finite and bounded piece of tape with ω, infinite, segments, each corresponding to an atomic proposition in the logic. Let the tape be inscribed with the symbol 0 or 1 according to whether the singleton set $\{n\}$ is a subset of a given subset of ω; alternatively, the segment $\{n\}$ of the tape is inscribed with a 1 iff $n \in S$ for a given subset $S \subseteq \omega$. For example, the finite subset of ω, $\{1,2,5\}$ corresponds to the tape inscription: -

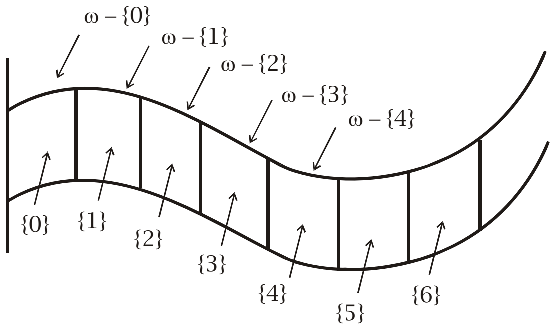


This corresponds to the lattice point of $\mathbf{2}^{\omega}$ given by $\{1,2,5\}$ and corresponds to the proposition $\alpha_1 \vee \alpha_2 \vee \alpha_5$. Here there are certain segments of the front of the tape that are undetermined. Suppose we fill in *all* the remaining segments with 0s.



This is *not* the same as $\{1,2,5\} \equiv \alpha_1 \vee \alpha_2 \vee \alpha_5$, because in it we have *definite negations*. It corresponds rather to $\neg\alpha_0 \vee \alpha_1 \vee \alpha_2 \vee \neg\alpha_3 \vee \neg\alpha_4 \vee \alpha_5 \vee \neg\alpha_6 \vee \ldots$ which is an infinite assignment of values to the front of the tape.

From $\alpha_0 \equiv \{0\}$ I may infer $\alpha_0 \vdash \alpha_0 \vee \alpha_n$ for any *n* by the rule for $\vee$-introduction, which is the rule of dilution of information. From $\alpha_0 \equiv \{0\}$ alone (without other negations), I may infer $\{0\} \vdash \omega$, which represents *total dilution* of the information contained in the atom $\alpha_0 \equiv \{0\}$. But I may not infer $\alpha_0 \vdash \neg\alpha_0$ because this is inconsistent. So if $\alpha_0 \to 1(\text{true})$ then $\neg\alpha_0 \to 0(\text{false})$. If I inscribe a 1 on any segment on the front of the tape I must correspondingly inscribe a 0 on the back of the tape; where the front of the tape is undetermined, so too is the back. If a segment on the front of the tape corresponds to $\{n\}$ and this is determined and we have $\{n\} \to 1(\text{true})$, then nothing else is determined (by this assignment alone) *except* that we cannot have $\omega - \{n\}$, for this is a contradiction. Therefore, if a segment on the front of the tape represents $\{n\}$ then the segment on the back of the tape represents $\omega - \{n\}$. Hence, the front of the tape represents finite singleton sets (atoms) and the back of the tape represents finite co-singleton sets (co-atoms); if we have $\{n\} \equiv \alpha_n$ on the front then we have $\{n\}' \equiv (\omega - \{n\}) \equiv \neg\alpha_n$ on the back; this may also be marked using the symbol for the complement $\alpha_n{}' \equiv \neg\alpha_n$. By combinations (unions) of singleton or co-singleton sets whose truths are represented by the front and back of the tape respectively, we see that the front of the tape collectively contains the set of all finite subsets of ω and the back of the tape contains the set of all co-finite subsets of ω.



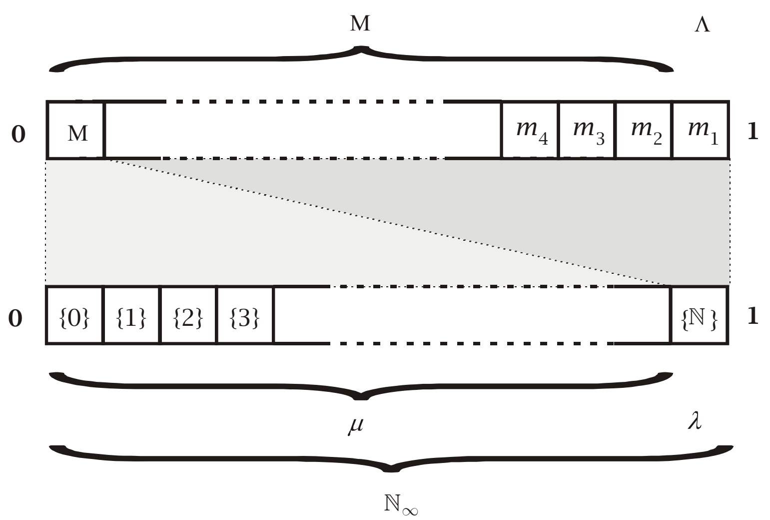
But since the tape contains an infinite partition of ω segments, assignments of 1s and 0s to *all* of these segments must also define those infinite subsets of ω that are not co-finite sets or intersections of co-finite sets; that is, the sets that belong to the boundary.

The issue of the boundary is problematic. So far as any segment of the tape is concerned, *although there are infinite segments* there is no actual segment of the tape that cannot be reached from 0 by a finite number of steps (counting up). This must follow from the fact that every member $n \in \omega$ is finite, and from the definition of ω as the *first* infinite ordinal. *Viewed from the 0 end of the tape*, the region "close to" 1 (its *neighbourhood*) appears as an infinitesimally small gap that is (a) inaccessible from the 0 end, and (b) contains an infinite collection of segments. This corresponds in our model to the distinction between $\mathbb{N}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\}$ $(\mathbb{N}_\infty = \mu \cup \lambda)$. Suppose from the 0 end one thinks one has identified a segment that is definitely within this inaccessible gap, then it is immediately discovered that this segment is accessible after all; no segment of the tape is actually inaccessible once it is determined to just which $n \in \mathbb{N}$ it corresponds; whenever an "observer" starting at 0 moves towards the gap, the gap appears to "shrink" away from that observer, and yet there always appears to be a gap that contain an infinite collection of inaccessible numbers. The picture is reversed from the 1 end of the tape. All of this follows from the observation that the underlying space defined by the segments of the tape is totally disconnected, so the neighbourhood of 1 is totally disconnected from the neighbourhood of 0 and appears to be an infinite distance from it, where distance is given by the metric defined by the number of segments separating any two segments. Between the neighbourhood of 0 and the neighbourhood of 1 *there is a boundary* that is corresponds to a boundary between the finite and infinite subsets of ω. I shall mark this boundary by (ω). It is a concept of an indeterminate separation between the finite and the infinite. In any physical application it can never actually be located, and any attempt to locate it immediately forces one to conclude that it lies somewhere else. Being indeterminate (ω) cannot be a set, for sets are determinate, definite multiplicities. [Chap.2 / 1.3.1] In the lattice generated by all subsets of ω, that is the Cantor space, $\mathbf{2}^\omega$ there is a correspondent boundary between the collection of all finite subsets of ω and the collection of all subsets of ω, finite or infinite. The collection of all finite subsets of ω is denoted by $F(\omega) \cong \mathbf{2}^{<\omega}$; which we have seen is an *ideal* within $\mathbf{2}^\omega$.

But the space $[0,1]$ under the infinite, actual division into ω parts is *totally separated*, which is as much as to say that *every segment whatsoever is separated from every other by a boundary*, so that an observer located at segment $\{0\} \leftrightarrow \alpha_0$ perceives that *every segment lies in a neighbourhood that could be interpreted as appearing to be an infinite distance away being separated by a boundary that cannot be reached.* This follows immediately if we adopt the principle that the only atoms that can be reached from any given atom $\{n\} \equiv \alpha_n$ are those to which this atom is connected. Then each $\{n\} \equiv \alpha_n$ is connected only to itself. (A topological space *X* is said to be *totally disconnected* if the only connected subsets in *X* are the one-point sets, and Brower's theorem that states that the Cantor set, $\mathbf{2}^{\omega}$, is totally disconnected. [3.12 above] The Cantor set inherits its totally disconnection from the partition of the underlying space which is also totally disconnected.) Consequently, there is a boundary separating any atom from every other atom.

4.1 (+) A principle of complementarity

We operate with *simultaneously* with two models that are not formally inconsistent with each other and represent different interpretations of the same underlying concept of an actually infinite partition of the real line $[0,1]$ into ω totally disconnected atoms. The validity of this principle was demonstrated above [Section 2.3 – Partition of the Cantor set] where the following diagram was constructed on the basis of the Stone duality: -

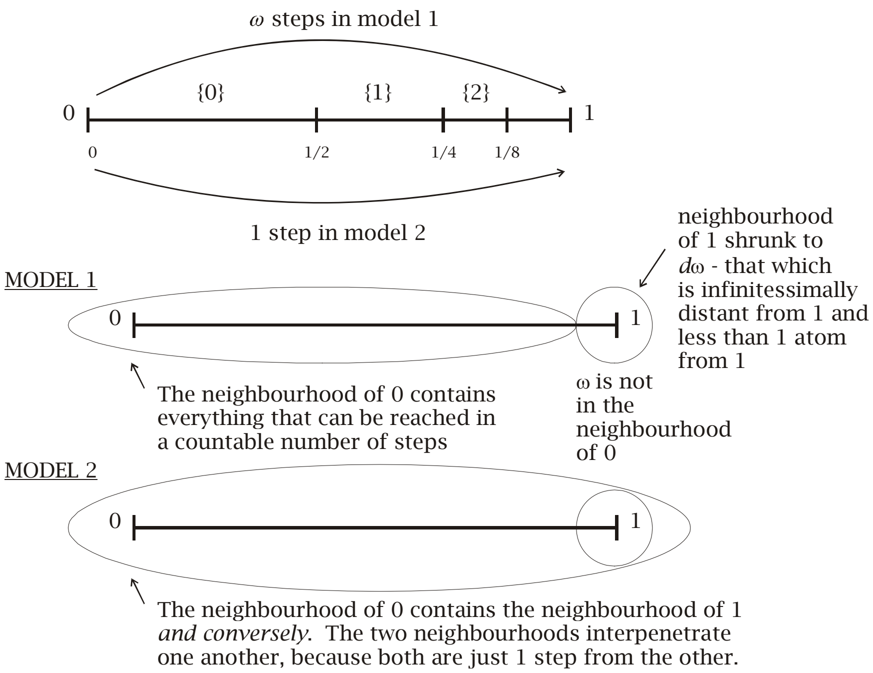


We see that from the perspective of 0, the interval $[0,1]$ is divided into two regions: -

1. Firstly, a region marked μ which comprises an unordered anti-chain of all the natural numbers $\mathbb{N}$. Each member of this region is an atom of the interval and lies at a distance of 1 unit in the metric from 0 when it is identified with the **0** of the Boolean lattice constructed over the partition.
2. Then there is a region marked λ which is also an atom in the algebra at a distance 1 unit from **0** and perceived from 0 to be the well-ordered chain, $\mathbb{N}$, *on which complete induction is defined.*

The region marked μ is not an atom of the interval, but the collection of all the atoms excepting that one marked by λ. Hence, while λ may be regarded as an atom, from the perspective of 0, μ is not an atom. Under the Axiom of Choice, which also calls λ into existence, μ may be regarded as a set that is a different description of the same collection as $\mathbb{N}$. We write $\mu = \mathbb{N}$. The two complementary models are: -

1. Every neighbourhood is separated from every other by a boundary that divides each into disjoint neighbourhoods so that the one cannot be reached from the other by any finite number of steps from atom to atom, but can be reached by a path of infinite discrete steps from atom to atom. This latter property follows from the principle that the line joining the two neighbourhoods can be partitioned into an actually infinite number, ω, of atoms. I shall call this *model 1.*
2. Every atom as just one step away from every other atom. I shall call this *model 2.*

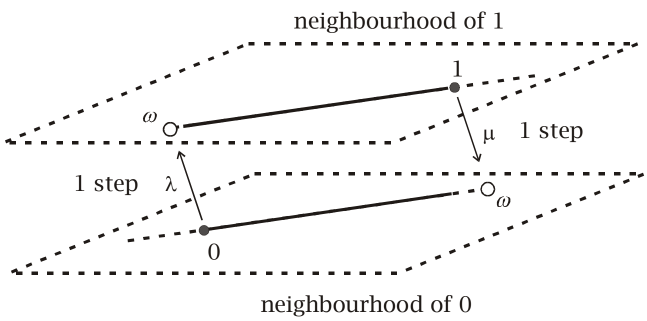


4.2 (+) Inverse steps between neighbourhoods

The segment $\{n\}$ is said to be *n*-steps away from the segment $\{0\}$ in the metric defined on the neighbourhood of 0. This metric is equivalent to *counting the number of atoms that separate two atoms.* The neighbourhood of 1 is also *one step* away from the neighbourhood of 0, and the neighbourhoods are said to be separated by *one atom* or at a distance of *one atom.* I denote the step from the neighbourhood of 0 to the neighbourhood of 1 by λ; conversely, the step from the neighbourhood of 1 to the neighbourhood of 0 shall be denoted μ. These steps are inverses of each other, so $\lambda = \mu^{-1}$.

4.4 (+) Two-sided sheet of paper model of Cantor space

One might argue at this point that the intuitive concept of a neighbourhood expressed in the above diagram is *inconsistent* and therefore *false*, but this is simply not true: certainly we can demonstrate the consistency of the concept by providing a physical model of this situation – that of *two sides of one sheet of paper.* On one side we have the neighbourhood of 0 and on the other the neighbourhood of 1: -



The model provided above provides some problems for point-set topology, but it is consistent in perceptual space. Let us image that both sheets of paper are infinite is size – that is isomorphic to Euclidean 2-space. Then we may say that the neighbourhood of 0 is contained in the neighbourhood of 1 and conversely, and yet the two neighbourhoods are not identical, because one side of a sheet of paper is not identical with the other. This situation contradicts the Schröder-Bernstein theorem. [Chap 2/2.7.5] However, as the two-sided sheet is a model of Cantor space, which is defined in point-set topology, it is not inconsistent with it.

The two sides of the sheet are isomorphic to E^2 and represent closed and bounded sets within themselves: E^2 as a subset of itself is closed. However, they are both open at the boundary between the two sides, which is *not an additional side or set*, since if it were there would be a boundary between the boundaries and so on ad infinitum (infinite regress). Therefore, the neighbourhood of 1 *closes* the neighbourhood of 0 and conversely. It is its one point compactification The neighbourhood of 1 is the closure and boundary of the neighbourhood of 0 and conversely and acts as its one-point compactifcation. [5.3.9 et seq.]

The mention of E^2 is as a heuristic only. In the context I am discussing the neighbourhood of 0 and 1 comprises a real line $[0,1)$ and $(0,1]$ respectively. This is isomorphic to the manifold $\mathbb{R}^+$. Over this manifold a *scaffold* (or skeleton) of $\omega \cong \mathbb{N}_\infty$ atoms is erected. E^2 helps us to visualise the relation that *a straight line has two sides separated by a boundary. The lines have no thickness, but yet can be represented as point sets; the boundary has no size and is incommensurable with the two sides of the line, so cannot be represented as a point set.* We can iterate the construction to create *many sides and many boundaries*; for example, addition of the neighbourhood of 2; however, the pairing of the two neighbourhoods is the essential structure from which the multiples would be constructed, at least so far as this context requires.

Under the principle of complementarity the neighbourhood of 0 comprises *both* $[0,1)$ and $[0,1]$ simultaneously. The steps λ and μ may now be regarded as *operations*. The boundary cannot be separated from a side, even if the two sides can be so separated *by a boundary*. What this means is that λ adds the boundary of $(0,1]$ and the whole of $(0,1]$ (the neighbourhood of 1) to $[0,1)$ (the neighbourhood of 0) and conversely for μ.

4.5 (+) Representations of the neighbourhood of 0

$[0,1]$ The neighbourhood of 0 comprises the whole space.

1 is a member of the neighbourhood of 0. It is 1 step away from 0. Every point in the neighbourhood of 0 can be reached in one step. The neighbourhood of 0 contains the neighbourhood of 1.

$[0,1)$ The neighbourhood of 0 comprises every point that can be reached in a countable number of steps in accordance with the scaffold. 1 is ω steps away and cannot be reached in 1 step. In this model the neighbourhood of 0 does not contain the neighbourhood of 1 as a subset.

$\{0\}$ The neighbourhood of 0 is everything that is less than one atom away from 0. $\{0\} \cap (0,1] = \varnothing$ - the neighbourhood of 0 and the neighbourhood of 1 are disjoint.

μ The boundary of $[0,1)$.

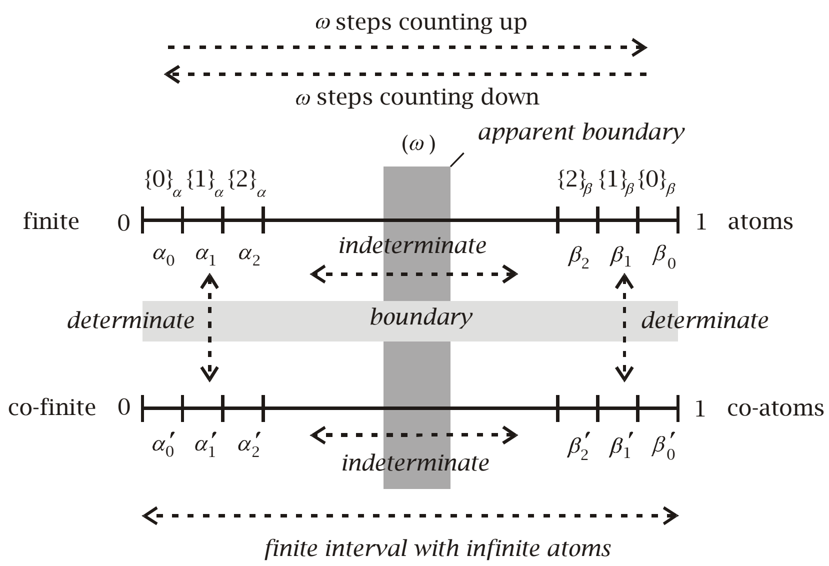
λ The boundary of $(0,1]$.

4.6 (+) Sizes of neighbourhoods

The size of the scaffold (skeleton) of $[0,1]$ is ϖ. Since $[0,1)$ is a proper subset of $[0,1]$ its scaffold must have a size smaller than ϖ. However, the size of this scaffold is larger than any finite number, although it *is* a finite number. It is a finite number larger than any finite number. Let us denote this number (ϖ). This number is a *limit point* between the finite and the infinite, and is not a natural number. It is not a real number either. It is an indeterminate number corresponding to the boundary between the finite and infinite partition of $[0,1]$.[4] All of this is consistent with the one-point compactification: $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\}$.

4.7 (+) Principle of symmetry

The assumption we are working under is of an infinite division of the line segment $[0,1]$ into actually ϖ indivisible pieces of information, or atoms. Counting these pieces of information (atoms) from 0 we reach 1 after ϖ steps, but it is impossible to determine the last countable number prior to reaching ϖ, which is undefined. Nonetheless, we may also reverse the picture, and starting at 1 count the ϖ piece of information *towards* 0. This follows by symmetry. This principle of symmetry follows from the Stone duality.

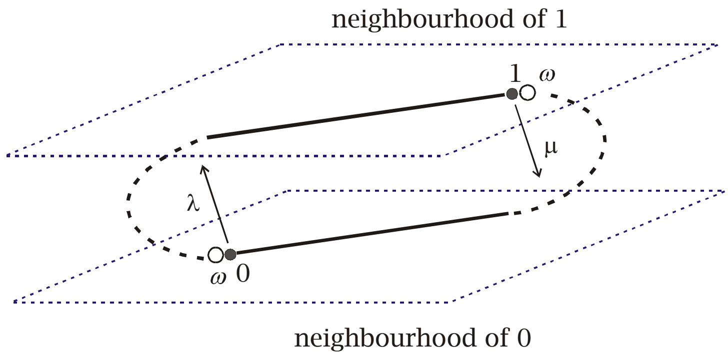


[4] Conjecture: I add the following conjecture, which I acknowledge I have not analysed further and may be sheer nonsense. The location of (ϖ) in any given interval $[0,1]$ is indeterminate. I conjecture firstly, that whenever we measure a particle located in this interval we give (ϖ) a temporary determination and that this temporary determination of the locus of the particle follows a probability distribution. I suggest that indeterminacy principles in physical applications may owe their origin to our system of determination of measure and coordinates rather than to properties inherent in their physical nature.

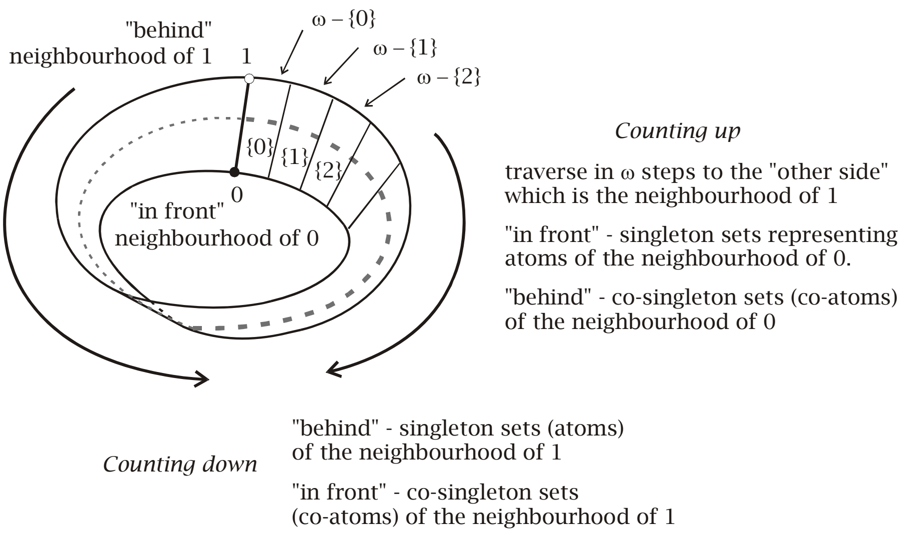
Any assignment of information (atoms) while counting up forces an assignment of information while counting down; so these two models are two models of the same information; they both represent the same partition of the interval $[0,1]$. Atoms that are a finite number of steps away from 0 are members of the *neighbourhood of 0*; and likewise, those atoms that are a finite number of steps away from 1 are members of the *neighbourhood of 1*.

4.8 The Möbius band model

The model of two sides of a sheet of paper in E^2 or a double-sided line is really only a picture of what I called "model 1" above – showing the neighbourhoods of 0 and 1 as disjoint and that 1 cannot be reached by a path of ω steps in the neighbourhood of 0 though it can be reached by a single "jump" of length 1. We seek a representation that combines both models of the neighbourhood. In this case we must incorporate the idea in "model 2" that 1 can be reached from 0 in ω steps as well as by a single jump. To achieve this we must connect the two neighbourhoods at both ends.



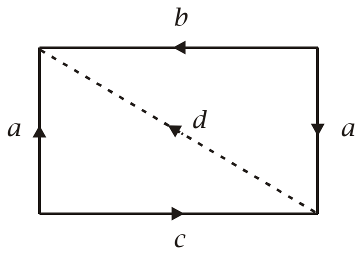
In terms of our counting up and counting down we obtain the following: -

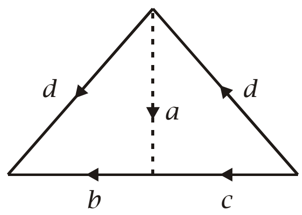


The result is not a cylinder because in model 2, the neighbourhood of 1 lies on the same side of the sheet of line (or sheet) as 0; therefore, the two sheets must be joined into a Möbius band, so we have a sheet of one face (side) only. Although the Möbius band is said to have only one face, a continuous one-sided curve from 0 back to 0 (passing through the neighbourhood of 1) makes two "loops". One loop is shown in thicker dots in the diagram.

4.9 (+) The Klein bottle model

If we allow ourselves to "write" on the "back" of the surface we can cut the Möbius band into two one-sided surfaces connected at a boundary, which is its edge. The canonical diagrams [Chapter 2 / 2.11.1 and Blackett [1982] Chapter 1] of the Möbius band are: -

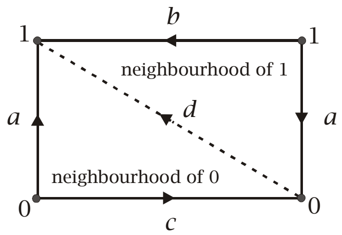


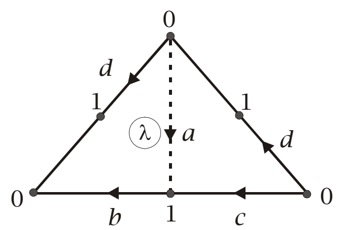


Which have equivalent algebraic edge equations

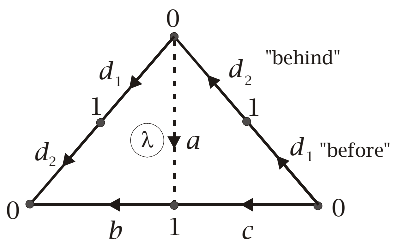
$abac^{-1} = 1$ $\qquad\qquad$ $dde = 1$

Inserting 0 and 1 as vertices into these diagrams gives

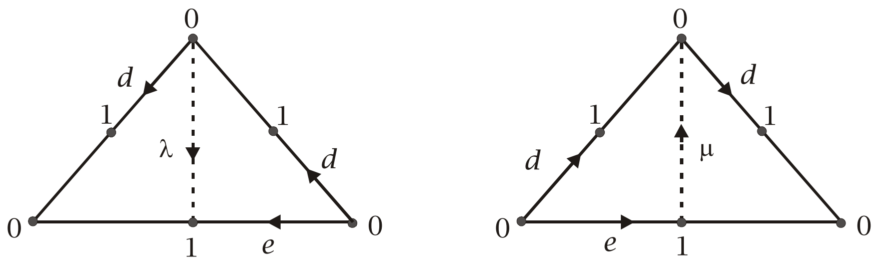




If in fact we cut a physical model of a Möbius band along the diagonal edge *d* as indicated in the diagram, then we obtain a triangular piece of material (paper) that is *two-sided*. The identification of the edges means that the two sides are identified and one side is continuous with the other in two directions. However, in the diagram above the vertices representing 1 are on the other side of the sheet to the vertices representing 0. The boundary edge here indicated by *d* may be subdivided into two parts – one part running from 0 to 1 lying on the "front" side of the sheet as it faces us and the second part lying on the "back" side.

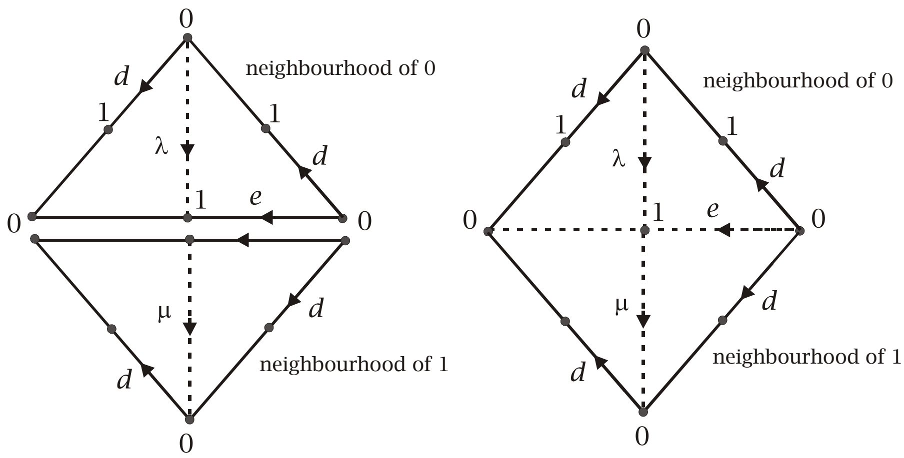


These are paths of ω steps that lie in the interiors of the respective neighbourhoods. The path d_1, for instance, lies wholly in the neighbourhood of 0. In this sense 1 is on the same side as 0. However, since both neighbourhoods interpenetrate each other both d_1 and d_2 lie on both sides of the sheets – they are two sided lines. On the other hand, the path labelled λ, displays the point 1 as in a disjoint neighbourhood to that of 0 requiring a one-step jump to the other side.



neighbourhood of 0 neighbourhood of 1

The direction of the arrows is reversed in the right-hand diagram because we have to take the sense (clockwise/anticlockwise) of the direction of the arrows. They are mirror images of each other.



This has algebraic edge equation: -

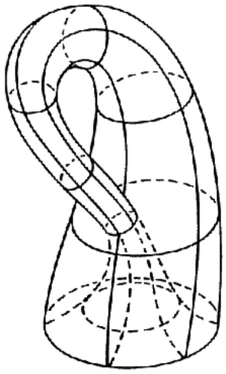
$ddd^{-1}d^{-1} = 1$

which reduces to: -

$dd^{-1} = 1$

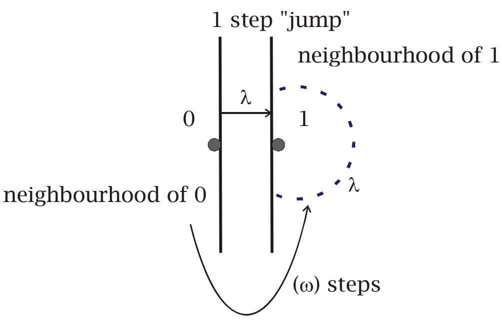
$1 = 1$

This does not define an allowable two-dimensional surface. However, this is because it is a model of a one-dimensional line or interval $[0,1]$. It does allow us to visualise the notion of interpenetrating neighbourhoods, and the equation $1=1$ shows that it is a consistent model of the unit interval. The above pictures are merely illustrations, and strictly there are no points in the space marked by the jumps λ and μ, so the fact that the diagram strictly "collapses" to $1=1$ is to be expected. Nonetheless, if we mark the edges *d* in the above diagram to indicate whether they lie in the neighbourhood of 0 or the neighbourhood of 1, we obtain the edge equation $dde^{-1}e^{-1}=1$, which is the edge equation of the Klein bottle. The Klein bottle is a model of the interpenetrating neighbourhoods of 0 and 1. The diagram also shows that λ and μ are inverses of each other.



As with the Möbius strip, a Klein bottle may have just one surface, but at any point the glass or substance from which it is made has a thickness and so each point on the surface is immediately associated with another point on the surface, and these two points form neighbourhoods that as "opposite" associates of the surface are disjoint, and yet may be connected by a continuous path. The material between the two "associates" is the boundary. Shrinking the boundary in the model to no dimension is equivalent to equating *d* with *e* in the edge equation $dde^{-1}e^{-1}=1$. This contracts the bottle to a single point. The disjoining of the associates is essential to the structure of the Klein bottle.

The connection from 0 to 1 on the Klein bottle, equivalent to taking one turn about the surface to arrive at the associated "opposite" side, so these are equivalent to paths of infinite length. Either we must allow the Klein bottle to be made of a sheet of *infinite size*, or we must surround each associate with a boundary representing the limit of the process of taking ω steps. So whether we "jump" from one "side" to the other, or pass continuously along the surface, we can only reach the one associate point from its pair by *taking a limit*.



A point set has a dimension; for example, the interval $\left[0,1\right]$ is one-dimensional. But if I take any *real surface*, for instance, that surface always has *two sides*. Now this concept of two sided figures does appear in algebraic topology in the distinction between orientable and non-orientable surfaces – the cylinder is 2-sided and orientable, whereas the Möbius band is 1-sided and non-orientable. But any point set describing the cylinder is a mapping from a 1-sided sheet of paper to a surface wherein the two-sided nature is not indicated. (In differential geometry a Monge patch always begins with a sheet and models a given surface as the image of that sheet, so the two-sided nature of a surface is never encoded. Whether a surface is orientable or not is not a property of two-sidedness but a global property of how the edges of a surface are connected. Even the paper from which one makes the "one-sided" and non-orientable Möbius strip *begins as a two-sided sheet.*) A point set of a cylinder does not indicate *which side of the cylinder you are on.* Now the Möbius band is 1-sided. However, the 1-sided nature of the band does not express the fact that a closed 1-sided loop on the band winds around any interior point twice; the band has a winding number. The winding number is not contained in the point set. So the winding number is extraneous information to the point set, not encoded in the enumeration of the point set; it is part of the *intension* of the Möbius band. Thus, the underlying assumption of set theory, namely that the content of every concept can be encoded by the extension of a set is *false*. *Intensions cannot be eliminated.* To illustrate this further, consider the following: -

A. A point set *X* coextensive with the point set of a Möbius band, together with the rule that any 1-sided loop in the band winds twice around some interior point; i.e. has winding number 2.

B. *X* as above, but with the rule that the winding number of any 1-sided loop is 1.

A indicates a *consistent* and *physically realisable* object; I suggest that *B* does not. However, the difference is not contained in the enumeration of the point set *X*, but in how the point set is *interpreted.*

Suppose we shrink the thickness of a sheet of paper to zero, and hence create an idealised sheet, as is the common practice in mathematics. Shall we say that we have an object with 1 or 2 sides? Before the idealisation, the sheet had 2 sides and a boundary separating the two. Now after the idealisation what is the number of sides of the sheet? The answer is that whichever model we adopt will depend on the use to be made of it *in context*. This is again an issue of how we are to *interpret* the model. The regular model interprets the sheet as a point set and therefore implicitly encodes the idealisation that it is a surface of 1 side. However, that does not preclude the possibility that an alternative encoding is needed in an alternative context, one that involves the idea of both sides. The same applies to lines inscribed on surfaces; sometimes they are 1 sided and sometimes 2. Indeed as a line can be the locus of the intersection of any number of hyperplanes, we may say that the number of sides of a line may be infinite.

CHAPTER 7

Proof paths and logical compactness

1 Embeddings

1.1 The story so far

Arithmetic is based on the set $\mathbb{N} = \{0,1,2,3,\ldots\}$, which is an unbounded domain of potentially infinite elements; it is equipped with a unique principle of reasoning, complete induction, that enables one to infer from finite premises to a conclusion about the set as a whole. In this way the collection $\mathbb{N} = \{0,1,2,3,\ldots\}$ is "completed", or, "the boundary is attained", because we have successfully inferred from *any* to *all.* An analytic logic cannot be defined directly over $\mathbb{N}$ for it forms a chain and the lattice that it generates is just another chain; furthermore, because $\mathbb{N}$ has no upper bound, the resultant lattice has no maximal point that can serve as **1** in an analytic logic. Arithmetic is not *prima facie* a form of analytic logic.

If arithmetic is a form of analytic logic after all, then the set $\mathbb{N} = \{0,1,2,3,\ldots\}$ must be embedded within a structure over which an analytic logic can be constructed. With this in mind, first-order set theory has been developed. In this theory a collection of entities known as ordinals is defined; these include transfinite ordinals. The ordinal that most concerns us is ω, which represents the notion of an actual, or completed infinity with the same members as $\mathbb{N}$, but now conceived as bounded into a set, which is a definite, determinate multiplicity; ω is said to be the first ordinal that follows in the sequence 0, 1, 2, 3, … and to be the first ordinal of infinite size, or cardinality. An analytic logic can be defined over ω as follows: imagine that the extended real line, $[0,1]$, is partitioned into ω segments that we shall call atoms. Although these atoms may be numbered and we may count them from 0 to 1 this primitive relation is ignored when we construct the lattice over it, for otherwise we shall not obtain an analytic lattice as required; the lattice that arises is the Cantor set, $\mathbf{2}^{\omega} \cong \{0,1\}^{\omega}$. In order to assess the claim that arithmetic is analytic, we must consider the possibility that all the theorems of arithmetic, including all those that depend on the natural order of $\mathbb{N}$ and principle of complete induction, may be derived from the analytic properties of this structure.

The Cantor set is subdivided into "regions" that correspond to its ideals and filters. One of these is the proper ideal of all finite subsets of ω. Under the Boolean Prime Ideal theorem (Boolean representation theorem), which in turn rests upon the Axiom of Choice, this ideal can be extended to a maximal ideal M with maximal element μ [Chap. 5 / 7.42]; likewise, there is a prime filter (ultrafilter) Λ with minimal element λ. Under the assumption of the Axiom of Choice, μ and λ become sets and we have $\mu = \lambda'$. The distance of μ from **1** is 1 unit in the metric, and λ from **0** is also 1 unit. Hence, the Axiom of Choice effectively transforms μ into a co-atom (prime) and λ into an atom of the complete one-point compactification, $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \{0,1,2, \ldots ,\mathbb{N}\} = \mu \cup \lambda$.

1.2 First-order theories in general

Analytic logic subdivides broadly into two parts: -

1. Pure analytic logic, which comprises the propositional and predicate calculus. Also, the predicate calculus with identity may be included in this category.
2. First-order theories in which a theory about some specific class of objects is said to be embedded in first-order logic; first-order logic is said to provide the underlying logic of the theory. Examples of this are first-order Peano arithmetic, first-order set theory and the first-order theory of rings.

First-order theories are defined by the addition of axioms to those of first-order logic. The axioms usually form a list that can be mechanically generated (recursively enumerated). It is also claimed that all mathematics can be written and conducted in first-order set theory, so that any other first-order theory could be regarded as a species of first-order set theory. First-order theories fall into two categories: -

1. Those theories that are "complete", or if they are not complete then could be completed.
2. Those theories that are "essentially incomplete".[1]

The questions that arise are as follows: -

1. How is the underlying lattice modified or altered when the proper axioms of a particular first-order theory are added? In particular, are new lattice points added when this happens? How, in terms of the lattice to which the logic is related, are the models of first-order theories related?
2. Why are some first-order theories complete and others incomplete, and what, in terms of the lattice, does this distinction entail?

[1] For technical explanation of this term see Monk [1976] chapters 13 to 16, and Tarski, Mostowski and Robinson [1953].

The axioms that define propositional logic collectively may be taken as a name of **1** in the correspondent lattice. [4.10.3] Their role is to define the structure of the lattice and hence they may be conjoined to any lattice point $\phi \equiv \mathbf{1} \wedge \phi$. It is a theorem that these axioms are realised in a lattice of just two elements; the prime indecomposable algebra $\mathbf{2} = \{0,1\}$. This is effectively a proof of the completeness [Chap.4 Sec.6] of the axioms, because it demonstrates that every lattice of which **2** is a factor is a lattice in which those axioms are realised. Since **2** is a factor of every (Boolean) lattice whatsoever, every lattice inherits those axioms: completeness indicates properties that are inherited by all lattices.

A formal predicate logic could be defined over a finite lattice, but in this case those lattice points corresponding to quantifiers could all be eliminated in favour of finite joins or meets (finite lattice points) so such a logic would be merely a propositional logic masquerading in disguise. Therefore, it is natural to consider that a formal predicate logic is one where there is at least a countably infinite collection of lattice points. In the Gödel-Henkin completeness theorem [4.1 above] the property of completeness is attached to all predicate logics; this theorem explicitly attaches this property to every denumerable lattice, which is the minimal lattice to which a non-trivial predicate logic applies. The proof of the theorem works by constructing a model – that is a lattice – out of the very terms of the language out of which the logic is itself constructed: a denumerable lattice takes the place of **2** in the predicate version of the completeness theorem, and acts as the minimum model. This model is not irreducible in the same sense in which the Boolean algebra **2** is irreducible, but it is a kind of factor nonetheless.

The effect of the addition of axioms Σ to those of predicate logic is a combination of one or more of the following: -

1.3 The addition of axioms

1. The axioms may force the lattice to contain a minimum number of lattice points; in other words, to affect its cardinality.
2. The axioms may define a lattice point within a lattice. A lattice point defines (a) an ideal or down set – every lattice point that falls in this down set is an instance of Σ. If ϕ is such a lattice point then $\phi \equiv \Sigma \wedge \phi$. All lattice points in the ideal correspond to models of Σ. Also we have (b) a filter $\Sigma \vdash \ldots$ of consequences of Σ - every member of the filter represents a structure that is in some sense a dilution of the structure defined by Σ.
3. The axioms may define second-order relations on the lattice. For example, the axioms of equality $(=)$ define relations on the filters contained in the lattice. [5.1.7]

1.4 Example, the axioms of number theory

In addition to those axioms that govern equality, the axioms of formal number theory include the following: -

$(S1)$ $x = y \supset (x = z \supset y = z)$
$(S2)$ $x = y \supset x' = y'$
$(S3)$ $0 \neq x'$
$(S4)$ $x' = y' \supset x = y$

These four axioms cannot be satisfied in a finite domain. They firstly imply that every element x of the domain has a successor x'; that is, $\vdash (\forall x)(\exists y)\, y = x'$[2], and this is impossible in a finite domain without circles, which are ruled out by $0 \neq x'$. This illustrates principle 1 above: The axioms may force the lattice to contain a minimum number of lattice points – to establish its minimum cardinality.

The possibility of *adding more axioms* to an existing theory creates a picture of the lattice being embedded in another larger lattice, and of these axioms defining both an ideal and a filter in that larger structure. Every Boolean lattice is capable of being embedded in a yet larger lattice, and the collection all Boolean lattices forms an unbounded collection – in other words, a proper class. Once we progress from first-order logic to a theory embedded in first-order logic we progress to a model in which the theory corresponds to a lattice point *not identical to* **1**, and hence defining both a filter and an ideal in the lattice. All lattice points that fall within the ideal correspond to *models* of the axioms. Every lattice point is the disjunction (join) of other lattice points. The lattice may or may not be atomic.

1.5 Example of embedding a theory within the lattice

Consider the theory of $\mathbb{Z}_2 \cong \dfrac{\mathbb{Z}}{2\mathbb{Z}}$. This is a particular ring. The members of this ring are equivalence classes $[0] = \{0,2,4,6,...\}$ and $[1] = \{1,3,5,7,...\}$. Its model is the set $\{[0],[1]\}$ conjoined with the ring axioms. This model is already based on a prior partition of the space $\mathbb{N}$ into these two equivalence classes. Then $[0]$ and $[1]$ stand for two mutually exclusive atoms, so $[0] \wedge [1] = \mathbf{0}$ and $[0] \vee [1] = \mathbf{1}$. (This is the Boolean algebra **2**.) Let us attempt embed this theory within the lattice defined by base partition of $[0,1]$ into $\mathbb{N}_\infty$ parts. Then $\{0,1\}$, $\{0,3\}$, $\{2,1\}$, $\{2,4\}$ all particular models of the theory – they are examples from an infinite list of sets. The model as a whole is *any one of these*, so we attempt to form the disjunction: $\{0,1\} \vee \{0,3\} \vee \{2,1\} \vee \{2,4\} \vee \ldots$, but this is not possible as it stands because this disjunction gives the **0** of the lattice. We must treat each member of the list $\{0,1\}, \{0,3\}, \{2,1\}, \{2,4\}, \ldots$ as a *new atom*, that is an ordered pair:

[2] $\neg(\forall x)(\exists y)\, y = x' \Rightarrow (\exists x)(\forall y)\, y \neq x' \Rightarrow (\forall y)\, y \neq a' \Rightarrow 1 \neq a' \Rightarrow 0' \neq a' \Rightarrow 0 \neq 0$

$(0,1)$. Hence, if we start with a partition of the lattice in which the atoms were represented by $\{0\},\{1\},\{2\},\ldots$, in order to interpret $\mathbb{Z}_2$ in this framework, we must *rewrite the atoms* in terms of more fundamental atoms: $(0,1),(0,3),\ldots$ and the model is the disjunction (join) of all of these. Let Σ stand for this join of atoms. This lattice point, Σ, has at least *two* different representations. (1) As the join, $\Sigma \equiv (0,1),(0,3),\ldots \vee \ldots$ of the atoms of the lattice that constitute all its individual models; (2) As the meet of certain axioms, for example, those governing the predicate calculus, those governing the ring theory and one concerning the size of the domain of the ring.

It would be as well to pause to reflect at this point how profoundly our picture of the lattice has now altered as a result of the change up from (a) first-order logic to general idea of (b) a theory embedded in first-order logic. The lattice corresponding to first-order logic is already a lattice based on a potentially infinite partition, but the axioms of the theory govern the whole lattice and conjointly are a name of **1** in the lattice. The lattice corresponding to a first-order theory must also be an infinite lattice, but a larger one[3] – the possibility of an actually infinite partition arises; the theory itself, denoted in general by Σ, represents a lattice point *somewhere in the middle* of the lattice. This lattice point is itself a join of infinite atoms[4], so it does not belong to the ideal of all finite subsets of the lattice and already represents a species of limit point in the lattice – that is, an infinite join. In order to reach this lattice we have had to *rewrite our atoms*. Thus, even if a lattice is atomic it is always possible to embed that lattice into another lattice with atoms that *lie below* those original atoms. A finite illustration of this phenomenon would be simply what happens when we step from the Boolean algebra $\mathbf{2}^2 \equiv (\mathbf{1},\mathbf{0},p,\neg p)$ to the Boolean algebra $\mathbf{2}^4 \equiv (\mathbf{1},\mathbf{0},p,\neg p,q,\neg q)$; we have *doubled the number of atoms*, so that combinations that appear to be atoms in $\mathbf{2}^2$ become joins in $\mathbf{2}^4$. (Compare p with $p \wedge q$.) What is an atom is relative to the partition of space – and when the partition is finer then the *atoms split*. What this reveals is that what is atomic in a lattice is not absolute.

1.6 (+) Definition, floor, lowering the floor, ceiling

A given atomic lattice has a set of atoms which shall be called the *floor* of the lattice. We progress up the lattice by means of the lattice algebra, or equivalently, by the logical operations defined over the lattice. But now we discover that there is an alternative route that takes us out the lattice to another lattice, a progression that is possible in the proper class of all lattices; we can progress *to different atoms*. This process shall be called *lowering the*

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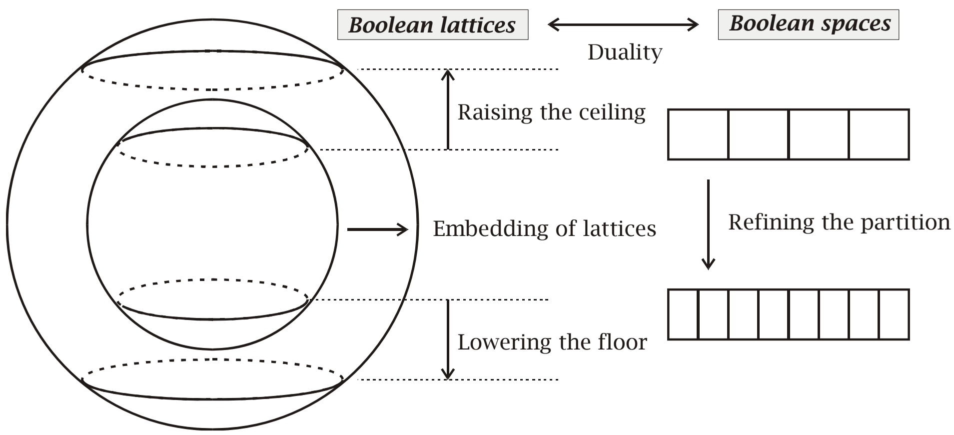
[3] I mean here "larger" in the sense of proper subset; the larger lattice may have the same cardinality as the smaller one.

[4] If the lattice is atomic; otherwise, it is generated from below by an infinite set.

floor. For example, when we rewrite the atoms of the lattice from $\{0\},\{1\},\{2\},...$ to $(0,1),(0,2),(0,3),...,(1,2),...$ we are *lowering the floor* of the lattice. We lower the floor when we progress from a model of $\mathbb{Z}$ to a model of $\mathbb{Z}_2$. The co-atoms of a lattice constitute its *ceiling* and when we lower the floor of the lattice we also raise its ceiling. Lowering the floor is a process of embedding a lattice within a larger lattice.



2 Proof paths and compactness

2.1 Semantic consequence

Let Γ represent a set of formulae of a first-order language *K*; suppose all of the formulae in Γ are held to be simultaneously true. Let ϕ be another formula of *K*. We write $\Gamma \vDash \phi$ if, in every model in which Γ is true, ϕ is also true. If a formula ϕ is true in every model we write, $\vDash \phi$.

2.2 Basic premise of the discussion

The basic premise of this discussion is that *every model is a lattice*. While it is possible to conceive of a general logic that has no direct relation to a lattice whatsoever (it is likely that the inferences we meet in natural language comprise logic in this sense), it is the very essence of formal, analytic logic to derive the properties of inference from the notion of an analytic partition of space into parts, and this intimately connects formal logic in this sense with the lattice so defined. We are testing the theory of effective computability – specifically, the theory that every mathematical proof is recursive. The domain of what is effectively computable is the countable, non-atomic lattice $\mathbf{2}^{<\omega}$ [See section 3 above]. Hence, if we deny that every model of a theory is a lattice we automatically refute formalism. For this reason, the discussion proceeds on the assumption that *every model is a lattice.*

On the principle that *every model is a lattice*, the relation $\Gamma \vDash \phi$ is interpreted as follows: Γ and ϕ are names of lattice points and $\Gamma \vDash \phi$ means ϕ lies in the filter generated by Γ. That is, to the language *K* there corresponds a lattice *L* and $\Gamma \vDash \phi$ in *K* corresponds to $\phi \in \text{filter}(\Gamma)$ in *L*. By an abuse of language we here use the same symbols in the language as for the filter, which is natural enough because even when we describe the filter directly *we use a language to do this*. Thus $\phi \in \text{filter}(\Gamma)$ also occurs in a *language*. However, in certain contexts it becomes essential to distinguish between the language, that is the formal logic *K*, and the lattice *L*. In that event we write $\overline{\Gamma} \vDash \overline{\phi}$ in the logic and $\phi \in \text{filter}(\Gamma)$ in the language.

The need for this distinction is that analytic logic, which is built over a lattice is created for the purpose of systematising *formal deduction*, which is a second relation not identical in meaning to consequence. The expression $\Gamma \vdash \phi$ is read, "there is a formal deduction of ϕ from Γ". When, for a language *K*, the notions $\Gamma \vDash \phi$ and $\Gamma \vdash \phi$ coincide the logic is said to be *complete*: "the completenesss theorem for logic becomes the assertion that there exists a notion of deduction, based on some clear cut, mechanical procedure for manipulating formulas of the language, such that $\vdash$ and $\vDash$ coincide." (Wolf [2005] p. 22.) The concept of formal deduction, $\Gamma \vdash \phi$, is associated with axioms and rules, and as the quotation from Wolf indicates there is an assumption that these axioms and rules are "mechanical procedures". It is implicit that such mechanical procedures are also effectively computable. These assumptions are questionable.

To say that a first-order theory is *incomplete*, symbolised by $\Gamma \vDash \phi$ but not $\Gamma \vdash \phi$, prima facie shows that *not all proof can be reduced to a mechanical procedure*. But as this is the very question at stake in this paper, let us not assume it, but rather seek to prove it.

A mechanical procedure must be actually finite. It may be arguably potentially infinite, but any actual proof, $\Gamma \vdash \phi$, must be a finite sequence of statements that start with Γ and conclude with ϕ. Smullyan writes, "By a *proof* in *A* [which denotes an axiom system] is meant a finite sequence $X_1, \ldots, X_n$ such that each term *X* is either an axiom of *A* or is directly derivable from one or more earlier terms of the sequence under one of the inference rules of *A*. A proof $Y_1, \ldots, Y_n$ in *A* is also called a proof of its last term Y_n and finally an element *X* is called *provable in A* or a *theorem of A* if there exists a proof of *X* in *A*." (Smullyan [1995] p. 80.)

In terms of a lattice, we interpret a lattice point Γ as being equivalent to the meet of those X_i in the proof $X_1, \ldots, X_n$ that are asserted contingently, and the remaining $X_j \in X_1, \ldots, X_n$ as defining a proof path in the filter generated by Γ. Then, the primary meaning of $\Gamma \vdash \phi$, in lattice terms, is that there is a finite proof path in the filter generated by Γ to $\phi \in \text{filter}(\Gamma)$. Of course, we must impose a rule that prevents the proof path from straying out of the filter – this is called *soundness*. Thus we have $\Gamma \vdash \phi \Rightarrow \Gamma \vDash \phi$ as the definition of a *sound proof*, meaning that there is a finite proof path in $\text{filter}(\Gamma)$ from Γ to ϕ. To say that a logic is complete, represented by $\Gamma \vDash \phi$ iff $\Gamma \vdash \phi$ means *primarily* that if ϕ lies in the filter generated by Γ, then there is a finite proof path from Γ to ϕ.

In a finite Boolean algebra and its concomitant propositional calculus the only kind of proof paths there can be are finite, and so we never have to consider anything other than a finite relationship in $\Gamma \vdash \phi$. When we step up to the predicate calculus we encounter quantifiers. Ostensibly quantifiers correspond to infinite meets and joins in a lattice. Suppose ϕ represents some infinite dilution of the information contained in Γ, that is, relative to Γ, ϕ is some infinite join, then we have a proof path that is infinite in length. A concrete example is the inference: -

$$\frac{Pa}{(\exists x)Px}$$

where the formula $(\exists x)Px$ represents an infinite join $Pa \vee Pb \vee Pc \vee \ldots$. Although this represents an infinite proof path, we still have a "mechanical procedure" joining *Pa* to $(\exists x)Px$ in the lattice, and the logic, in this respect at least, remains complete. Therefore, we must extend the notion of "finite" to encompass "mechanical procedures" that are notionally infinite but *like finite proof paths in all essential respects*. As one can see in this example, denoting *Pa* by Γ and $(\exists x)Px$ by ϕ, and assuming $(\exists x)Px$ represents an infinite join, then Γ lies "way below" ϕ in the lattice. Nonetheless, the expression: -

$$\frac{Pa}{(\exists x)Px}$$

is the clue to this situation. Although the distance in the metric between Γ and ϕ is infinite, the rule of inference allows one to "step over" this infinite path and reduce it in one go to a single finite step. In terms of a lattice homomorphism, Ψ, this is a mapping from the lattice in which $\Gamma \vdash \phi$ is an infinite path to another lattice in which $\Psi(\Gamma) \vdash \Psi(\phi)$ is finite. When this is possible we say that the logic is *compact*.

2.3 (+) Definition, compact proof path

A proof path $\Gamma \vdash \phi$ is said to be compact if whenever there is an infinite sequence in Γ then this has a finite subsequence.

Compactness (in formal analytic logic) is usually defined as follows: -

2.4 Logical Compactness Property

If every finite subset of a set of sentences Γ has a model, then Γ has a model.

I need to demonstrate the connection between this definition and the one I have presented above in terms of compact proof paths. Before I do so I must comment again on the ambiguity in the meaning of the symbol Γ in the expression $\Gamma \vdash \phi$. Γ of course stands for a set of propositions, but a set is usually a *disjunctive list*:: -

$\{S_1, S_2, S_3\} = \{S_1\} \cup \{S_2\} \cup \{S_3\} \equiv S_1 \vee S_2 \vee S_3$

whereas Γ is usually interpreted to be a *conjunctive list*. This is the interpretation that is consistent with all the usual definitions are cast. For example, Smullyan's definition of satisfiability: "A formula *X* is called (truth-functionally) *satisfiable* iff *X* is true in at least one Boolean valuation <u>in which every element of *S* is true</u>. Such a valuation is said to *satisfy S*." (Smullyan [1995] p. 11.) My underlining. (Note, to say a set of sentences is satisfiable means that it has a model; this also entails that it defines a filter within a lattice.) This requires $\{S_1, S_2, S_3\}$ has a model $M \equiv (S_1 \text{ is true in } M) \wedge (S_2 \text{ is true in } M) \wedge (S_3 \text{ is true in } M)$. It is a conjunctive set. Now suppose Γ fails to be logically compact, then it represents an infinite meet and the path from Γ to any consequence of all of Γ, $\Gamma \vdash \phi$, must be infinite in length because every element of Γ is required to make it. That is, let $\Gamma = X_1, \dots, X_n, \phi$ be an infinite proof of ϕ; then if $\Gamma \vdash \phi$ is not compact no finite subset of Γ suffices to prove ϕ.

Similarly, to show that a set Γ is not compact, then every finite subset of Γ has a model but Γ does not have a model. From Boolos and Jeffrey [1980] (p.141) we have a (second-order) example of a non-compact set. Let Γ_0 represent the axioms of second-order Peano arithmetic and $\Gamma = \Gamma_0 \cup \{X_n\}$ where $X_n \equiv a \neq n$. Then any finite subsequence of Γ is satisfiable, and hence representing a finite proof path in the filter generated by $\Gamma_0 \equiv \mathbf{PA}$, but no infinite sequence is satisfiable. This means that a sentence is implied by a set of sentences iff it is implied by some finite subset of it.

Thus the logical compactness property and the compact proof path are just variations of the same underlying fact. Likewise, the notion of a compact proof path is related to the classical topological definitions of compact space and sequential compactness.

2.5 Result, equiavalence of logical and topological compactness

Let $\Gamma \vdash \phi$ where $\Gamma = X_1, \dots, X_n, \phi$ is an infinite proof of ϕ in a Boolean space *B*. All lattice points of a Boolean space are clopen sets, hence the collection $C = \{X_n\} \cup \phi$ is an open cover for the subspace $\Gamma \subseteq B$. If $\Gamma \vdash \phi$ is compact then there exists a finite subcover of *C*.

2.6 Proving logical compactness and completeness

One route to proving that an analytic logic *K* has the compactness property is as follows: -

2.7 Proof of the logical compactness property

<u>Logical Compactness Property</u>

If every finite subset of a set of sentences Γ has a model, then Γ has a model.

<u>Lemma</u>

If Γ is inconsistent, then some finite subset of Γ is inconsistent.

Proof of compactness

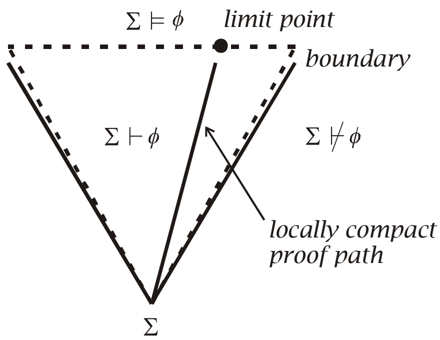
1. Assume that every finite subset of Γ has a model.
2. By the completeness theorem, if Γ is consistent, then Γ has a model.
3. From (1) and the lemma, Γ is consistent.
4. From (3) and (2), Γ has a model.

The proof indicates that the compactness of a logic K follows from its completeness: $\Gamma \vDash \phi \Rightarrow \Gamma \vdash \phi$. Hence *completeness is the more fundamental property*. We can summarise the above theorem by: complete $\Rightarrow$ compact. By contraposition a space (lattice) that is not compact is incomplete.

The paradigm of a complete lattice is any finite Boolean algebra. Every filter of such an algebra is finite, hence in the logic built over such a lattice (propositional logic) it follows automatically that $\Gamma \vDash \phi$ iff $\Gamma \vdash \phi$ (provided the deduction relation is sound).

2.8 Incompleteness

Incompleteness is $\Gamma \vDash \phi$ but not $\Gamma \vdash \phi$. Let us interpret this in terms of a lattice. The relation $\Gamma \vdash \phi$ is based on a notion of compact proof path; if $\Gamma \vdash \phi$ is infinite there is a finite sub path. Hence we can assume that $\Gamma \vdash \phi$ is effectively finite. In a formal analytic logic, this also means that $\Gamma \vdash \phi$ is effectively computable. Incompleteness requires that the filter $\Gamma \vDash \phi$ contains essentially infinite proof paths – in other words, proof paths that are not compact. The model that is envisaged is one of a filter defined by a set $\Sigma \subseteq \Gamma$ which contains non-compact infinite proof paths. This means that there are limit points in the filter that cannot be reached by the finite (i.e compact) proof paths starting at Σ. Let the set of limit points of a filter $\Sigma \vDash \phi$ be called its *boundary*.



A lattice may be locally compact without being complete, and this is precisely the case of the lattice all finite and co-finite subsets of ω, denoted $CF(\omega)$. $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$. $\mathbf{2}^{<\omega}$ is locally compact, and so too is $CF(\omega)$; but although composed of only clopen sets $\mathbf{2}^{<\omega}$ is *potentially infinite*, unbounded at its boundary. What this means here is that any given model of $\mathbf{2}^{<\omega}$ is a lattice of actually finite size, $\mathbf{2}^n$, $n \in \mathbb{N}$ and capable of being embedded in a larger set $\mathbf{2}^{n+1}$; another way of describing this is that $\mathbf{2}^{<\omega}$ is a *proper subset* of itself: $\mathbf{2}^{<\omega} \subset \mathbf{2}^{<\omega}$. This is possible because $<\omega$ does not represent a number in the usual; if it is a number then it must be an *indeterminate one*. In any practical situation the indeterminate $(\omega) = (<\omega)$ must be "sampled" – that is *determined*, and hence given an actually finite size, *n*. Then one determination of the indeterminate may always been included as a proper subset in another; hence $(\omega) = (\omega)$ is not a law that applies to $(\omega) = (<\omega)$.

The precise way in which this "openness" of $\mathbf{2}^{<\omega}$ at its boundary is reflected in its formal properties are as follows: -

1. $\mathbf{2}^{<\omega}$ is incomplete. There are joins and meets of elements of $\mathbf{2}^{<\omega}$ that are not contained in it.
2. Any *sufficiently strong* [Chap. 9 Sec. 1.6] formal analytic logic, *K*, that is built over $\mathbf{2}^{<\omega}$ is incomplete: there exist lattice points $\Gamma, \phi \in \mathbf{2}^{\omega}$ (note change to Cantor set) such that $\Gamma \vDash_K \phi$ but not $\Gamma \vdash_K \phi$.

The language, *K* permits the definitions of filters in $\mathbf{2}^{<\omega}$ that indicate the existence of lattice points that do not belong to $\mathbf{2}^{<\omega}$. When we say, "There are joins and meets of elements of $\mathbf{2}^{<\omega}$ that are not contained in it." we are already discussing the properties of the lattice $\mathbf{2}^{<\omega}$ in a meta-language, so have strictly speaking *gone beyond what it is possible to say about* $\mathbf{2}^{<\omega}$ from within $\mathbf{2}^{<\omega}$. When we add a formal language *K*, we merely formalise this informal argument already taking place within the metalanguage. Using the flatlander metaphor[5], a dweller living inside $\mathbf{2}^{<\omega}$ could never know that his world was incomplete. Even if he keeps enlarging his lattice ("world") by the process of lowering the floor [1.6 above and Chap.5 / 5.8] he could move towards the boundary *for ever* and never reaching it, never know that there was a boundary. The boundary will always some infinite distance away from him.

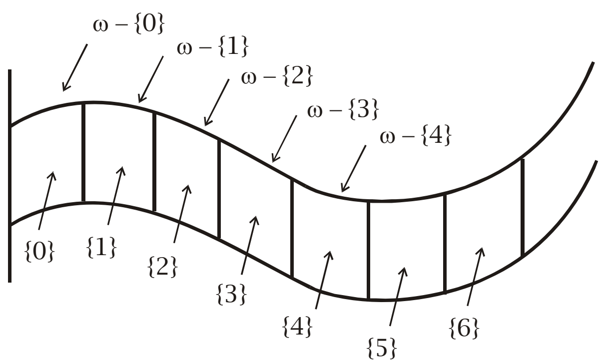
3 The domain of the effectively computable

There have been many analyses of what effectively computable: -

1. Turing analysis
2. Recursive functions
3. Markov algorithms
4. Abacus machines
5. Post systems
6. Lambda calculus

[5] A two-dimensional world where "inhabitants" are unable to imagine a three-dimensional one.

They have all been shown to be equivalent [See Chap.2 / 1.2 for discussion], and hence we may at any stage of our argument adopt any one of these as an analysis of any other – switching as the case requires between the different representations of the same underlying notion. Nonetheless, of these differing representations of effective computability, the Turing analysis has some claim to being the most fundamental, since it is also the most conceptual. As is well known, the Turing analysis is based on the idea of a machine moving along a tape that is partitioned into segments and can carry symbols upon it. The machine, in accordance with a deterministic program, can alter the symbols of the tape and move along it. It has been shown that any program operating with a collection of *n* symbols is equivalent to one operating with just 2 symbols, and here we take the set $\mathbf{2} = \{0,1\}$ as the basic set of symbols for the tape [See Chap.2 / 2.2.2]. There is an *intimate* connection between this description of a machine and the architecture of a digital computer. Digital computers are composed of *binary switches*. The basic binary switches are the NOT, AND and OR gates, and from these every computable function whatsoever is composed as a Boolean-valued function of inputs from $\mathbf{2}^n = \{0,1\}^n$. A consistent, sound computation is one that moves within a filter defined by some input $S \in \{0,1\}^n$. It is self-evident that the domain of any effective computation whatsoever is <u>at most</u> $\mathbf{2}^{<\omega}$ or any Boolean sum of it. It is usual to regard the Turing tape as based on an infinite partition of an interval into segments. But here we must be very careful in the description. <u>It is never possible to partition an interval (or tape) into an actually infinite number of segments</u>, so the maximal domain of a computer is never actually infinite, only potentially so. Indeed, we should at this moment remark that *in fact* no actual computer is even potentially infinite. The largest computer imaginable is always just a large, <u>but finite</u>, machine whose domain is *at most* $\mathbf{2}^N$ for some "large" $N \in \mathbb{N}$. While the term "large" here is used, it is a very subjective term, and means large relative to some subjectively "small" number; however, when comparing any "large" number $N \in \mathbb{N}$ with ∞, N is always "infinitesimally" small; no number $n \in \mathbb{N}$ whatsoever is ever commensurable with the potential infinite. The following illustration of the Turing tape: -



displays it as an isomorphic copy of $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$. On the front of the tape there are the atoms of the lattice ideal of all finite subsets of ω and on the back the co-atoms. A finite subset of ω is a configuration of the front of the tape; such a configuration automatically determines a configuration of the back of the tape. Each segment, as indicated, corresponds to an atom; for example, the configuration $1\,1\,0\,1\,0\,0 \ldots$ corresponds to the partition $\{0\} \cup \{1\} \cup \{3\} = \{0,1,3\}$. Let the atomic proposition in the logic built over this lattice be $\alpha_n \leftrightarrow \{n\}$, then this configuration corresponds to the proposition $p \equiv \alpha_0 \vee \alpha_1 \vee \alpha_3$. It is to be noted that *every actual Turing machine* has a maximal size of information it can manage, which corresponds to a definite partition of the tape. Thus, once the tape is partitioned into notional atoms, this constitutes a floor to the lattice, and it is not possible on the tape to compute the meets of atoms; for atoms $\alpha_i, \alpha_j, i \neq j$, the meet $\alpha_i \wedge \alpha_j$ does not belong to the lattice and cannot be effectively computed. It can be designated meaningfully in the language K built over this lattice, and we see automatically that *the language is always of greater expressive capacity* than the tape over which it is built and whose properties it describes. Of course, we can always *lower the floor*; in practical computing terms this amounts to either reconfiguring the tape so that each notional atom is split into others, or building a new and larger machine. This *potential* for enlarging the machine is also reflected in the usual assumption about the Turing tape that it is infinite in both directions. Also, theoretically we could interpolate any finite piece of tape between any two given segments; but this is a theoretical possibility only, because in practice the domain of any real machine is never $\mathbf{2}^{<\omega}$ but actually $\mathbf{2}^N$ for some "large" $N \in \mathbb{N}$.

Consider a specific example of a generalization that can only be obtained in number theory by complete induction; the simplest will do: -

$$(\forall n)\left(1 + 2 + \ldots + n \equiv \frac{1}{2}n(n+1)\right)$$

Interpreting this in terms of the lattice is problematic because this involves the relationship between logical and algebraic operations. Of course, it is well known that the recursive algebraic operations, for example those based on addition and multiplication, can be effectively computed; but their relation to the logical inferences is complex. For example, let a Turing tape be inscribed with the symbols: -

$0 \mid 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; \ldots$

It is easy to write a program that will replace this string by

$0 \mid 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots$

But let this be interpreted in logical terms, where the symbols represent atomic propositions, then the program computes: -

$\alpha_1 \vee \alpha_2 \vee \alpha_3 \vee \alpha_4 \vee \alpha_5 \vdash \alpha_1$

which is an unsound inference. What this essentially illustrates is that taken as a whole unrestricted effective computing is inconsistent, and in practice there must be a huge mental effort required to prevent actual programs from computing inconsistent results. It is well known, for instance, that Church's lambda calculus is inconsistent, and restrictions have to be placed on its use in practice.

Nonetheless, the algebraic and arithmetic operations, addition and multiplication, are effectively computable in terms of Boolean functions, so this raises the murky issue of the relations between the two systems. Rather than get bogged down in these we can adopt a simple set of rules: (1) Let there be two Turing tapes and a Turing program for exchanging information between them. (2) Let one of these tapes be reserved for purely logical inferences representing dilutions up the lattice; let the other tape be a "scratch pad" where algebraic and arithmetic operations are computed "on the side". These two rules circumvent the need to precisely describe the domain of <u>consistent</u> effective computing because – even if the program is inconsistent, then we can never write down an actually infinite list of 1 s on the Turing tape, so the computer is restricted to the analytic compact part of the lattice, any part that is isomorphic to $\mathbf{2}^{<\omega}$ and no more. The program cannot literally enter the boundary region; and if it is claimed to compute any result appertaining in denotation to that region, then it does so <u>as a simulation</u>, and this process is strictly a compact and finite one.

The subject of atomic lattices is central to any discussion of the limitations of effective computability. A lattice that has an enumerable list of notional atoms may be regarded as effectively computable because we can build up any formula mechanically from the "bottom" that is from the set of atoms; if the list is *finite* then this is not in dispute. If the enumeration is countably infinite, then this is disputable, because *as a matter of fact* every computer is in truth a finite machine, with finite processing capabilities and "memory", so the domain of a computer is *not infinite*. This is an *error* in the usual Turing analysis of effective computability because Turing assumes machines that have countably infinite memory. However, this assumption of Turing's is based on an implicit *argument by complete induction* – namely, that if we have a machine with finite capacity indexed by some ordinal n then we could *in principle* build a machine with finite capacity $>n$. This is the usual vicious circle [Chap.1 Sec. 7] in the formalist position. Yet, for the sake of argument, I will, in agreement with Turing, take the domain of the machine to be the potentially infinite. Let us accept, then, ∞ as the *upper limit* on an effectively computable process. Note, this is not the same as saying that the domain of an effective computation is ω, since that is a completed, actual infinity, and no machine can ever complete an enumeration.

Both sources of failure important. Whenever a proof path fails to be logically compact then there is a *prima facie* failure of effective computability. We can no long build the formulae up "from the bottom" and if a theory is built over a non-atomic lattice then it is *prima facie* not effective. However, that does not quite settle the matter because it may yet be possible to argue that there is an effective *simulation* [See 13 / 2.2] of the non-atomic lattice. There is an effective simulation when there exists a structure preserving function that maps back the non-compact inferences into the notionally atomic region: $\mathbf{2}^{<\omega}$.

It is a theme of this paper to point out the systematic failure in the literature to distinguish between the potential and the actual infinite. This error is crystalised in the assumption that one may write $\mathbb{N} = \omega$, which is a paralogism equivalent to saying that the potential infinite ∞ is an actual infinity and that we can finish counting the natural numbers. I am not claiming that ω is a meaningless concept; on the contrary it is the basis on which the whole discussion of set theory and logic in general proceeds; what I am pointing out is that as a concept it is by no means the same as the concept ∞. Furthermore, this is no mere theoretic distinction but a concrete reality. We can never actually finish counting the natural numbers; no machine can ever actually complete an infinite tally; there is no machine with an actually infinite number of internal states, and so on and so forth. Furthermore, we may even regard ω as the 1-point compactification of $\mathbb{N}$ and write $\omega \cong \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. When dealing with the Turing tape the distinction between ∞ and ω should be preserved. In this case we may say that the Turing tape is indeterminately bounded above, and we may represent this notion by (ω); this symbol (ω) effectively takes the place of ∞ in ω, that is when we take the 1-point compactification of $\mathbb{N}$, (ω) appears within $\omega = \mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ as the indeterminate upper limit of the sequence of natural numbers 0, 1, 2, ... that can never be reached by counting up alone.

Whenever we seek to determine (ω) we translate it into an actual number N and then immediately discover that the “real” (ω) lies somewhere else above N.

This image expresses the whole error of formalism: the determination of a concept results in a formal image of that concept equivalent to a natural number $N \in \mathbb{N}$ but never actually embraces the whole meaning of that concept, which slips away from it; here the concept in question is that of the potential infinite; regarding the thesis of artificial intelligence, we may say that human beings understand concepts and computers understand nothing – they compute.

As indicated above, we can interpolate into the domain of the computer $\mathbf{2}^{(\omega)} = \mathbf{2}^{<\omega}$ other copies of $\mathbf{2}^{(\omega)}$; and we may continue this “ad infinitum”. That is: -

$$\mathbf{2}^{(\omega)} \oplus \mathbf{2}^{(\omega)},\ \mathbf{2}^{(\omega)} \oplus \mathbf{2}^{(\omega)} \oplus \mathbf{2}^{(\omega)},\ \dots\ (\omega)\cdot\mathbf{2}^{(\omega)},\ \dots,\ \left(\mathbf{2}^{(\omega)}\right)^{\mathbf{2}^{(\omega)}},\ \dots,\ \left(\mathbf{2}^{(\omega)}\right)^{\left(\mathbf{2}^{(\omega)}\right)^{\left(\mathbf{2}^{(\omega)}\right)^{\mathbf{2}^{(\omega)}}}},\ \dots$$

(This sequence is based on ordinal arithmetic.) This sequence is bounded above by (ω).

4 Proofs of the completeness of first-order predicate calculus

There are "constructive" and "non-constructive" proofs of the completeness of first-order predicate logic – and examination of both shall be useful for us. By "constructive" in this context I refer to those proofs that actually provide a procedure for testing the validity of any possible formula – they give a method of positive validity. By "non-constructive" in this context I mean that the proof of completeness shows only theoretical completeness without demonstrating, given $\Gamma \vDash \phi$ how to obtain the finite proof: $\Gamma \vdash \phi$. I shall begin by examining the "non-constructive" Gödel-Henkin proof of completeness.

4.1 The Gödel-Henkin proof of completeness

<u>The starting point</u>

The starting point is the theory of the predicate calculus, *K,* which is here represented by a (conjunctive) set of axioms Σ_0, which are the axioms of the predicate calculus, together with its formal rules. The axioms may be adjoined to any lattice point whatsoever, so that if ϕ is a logicaly valid formula and we have $\vdash \phi$ then we may write $\Sigma_0 \vdash \phi$. Any proper filter within the lattice defined by these axioms constitutes a first order theory. We denote an arbitrary filter by Σ. A filter is the conjunction of notional atoms $\bigwedge \alpha_i$ for some index set $i \in I$ together with Σ_0. $\Sigma = \Sigma_0 + \bigwedge \alpha_i$.[6] The bulk of the proof takes place within two lemmas, here referred to as lemma [1] and lemma [2].

<u>The framework of the proof</u>

To prove: $\Sigma \vDash \phi \;\Rightarrow\; \Sigma \vdash \phi$, which is completeness. The proof is by contraposition.

Suppose	$\Sigma \nvdash \phi$	
Then	$\Sigma + \neg\phi$ is consistent	By lemma [1]
Then	$\Sigma + \neg\phi$ has a model	By lemma [2]
Hence	$\Sigma \nvDash \phi$	

This follows because if $\Sigma \vDash \phi$ every model of Σ must be a model of ϕ, but by step [2] there is at least one model which is a model of Σ but not of ϕ.

Therefore	$\Sigma \nvdash \phi \Rightarrow \Sigma \nvDash \phi$
Hence	$\Sigma \vDash \phi \;\Rightarrow\; \Sigma \vdash \phi$

So this outline indicates that the framework of the proof is relatively straightforward. However, the proof as a whole is intricate. This is because of the two lemmas. The first of

[6] The + sign here is used by Crossley et al [1972]. In view of what I have observed, its use is essential. The union sign is <u>not correct</u>. We are adding axioms. However, by this token it is not correct to view Σ as a set either!

these is relatively simple; it is the second that is "difficult" and in turn rests on other lemmas, so this is a multi-stage proof. In essence we are working backwards through the lemmas, proving them in reverse order so that the general outline of the proof is not lost.

Lemma 1

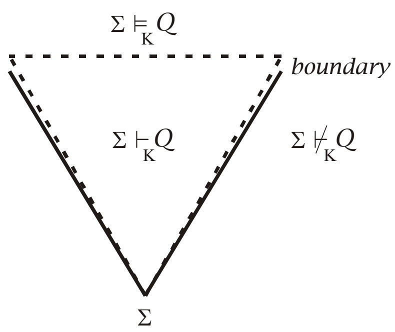
$\Sigma \nvdash \phi \Rightarrow \Sigma + \neg\phi$ is formally consistent.

4.2 Critical observation

The addition of the term "formally" to this statement is my own. I have added it as a result of analysis of the difference between this proof and the proof of the incompleteness of arithmetic in Gödel's incompleteness theorem. Here let us designate first-order arithmetic by K_0. (We use *K* for the predicate calculus and K_0 for arithmetic.) The theory K_0 is an example of a theory that is *sufficiently strong* – a concept that I shall formally define subsequently. [Chap. 9 / 1.6] Then we have the following "paradox" that needs resolution:-

1. By the Gödel-Henkin completeness theorem "all" theories of predicate logic are complete.
2. By Gödel's completeness theorem K_0, which is a first-order theory embedded in first-order predicate logic, is incomplete. Also, any sufficiently strong first-order theory is incomplete.

The puzzle is resolved by an appropriate interpretation of what the term "all" in the first of these statements means. All first-order theories contain analytic sub-domains that are isomorphic to the domain of pure first-order predicate logic, and these sub-domains are complete – and since compactness follows from completeness – they are compact. This sub-domain shall be shown to be isomorphic to $\mathbf{2}^{<\omega}$, and so the completeness theorem is consistent with the incompleteness theorem because it says $\mathbf{2}^{<\omega}$ is a factor of any model of a sufficiently strong first-order theory. This is in the sense of a quotient algebra – that is to say, any sufficiently strong first-order theory is a product of lattices isomorphic to $\mathbf{2}^{<\omega}$. The natural model of analytic logic is the Cantor set, $\mathbf{2}^{\omega}$, and we have seen that $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ is a quotient algebra of $\mathbf{2}^{\omega}$. [5 / 5.13] When we examine Gödel's theorem we will discover that there is a sentence *Q*, called the "Gödel sentence" that is undecidable; that is, neither $\Sigma \vdash_{\mathrm{K}} Q$ nor $\Sigma \vdash_{\mathrm{K}} \neg Q$ for all Σ. (The subscript K indicates that the lattice is sufficient large to reflect the theory of arithmetic. In this context K_0 represents the proper axioms of arithmetic; Σ_0 the axioms of predicate calculus, and $\Sigma = \Sigma_0 + \mathrm{K}_0 + \bigwedge \alpha_i$ where α_i are atoms is a filter.) Now *Q* is "true" in K , so that we have $\Sigma \vDash_K Q$, hence $\Sigma + \neg Q$ does not have a model – yet $\Sigma + \neg Q$ is not formally inconsistent in the sense that it does not imply a contradiction. In a manner of speaking $\Sigma + \neg Q$ is an inconsistent "set" that cannot be proven *within* K to be inconsistent. This is illustrated by the following diagram: -



Proof of lemma [1]
Suppose the statement
$\Sigma \nvdash \phi \Rightarrow \Sigma + \neg\phi$ is consistent
is false, then $\Sigma \nvdash \phi$ and $\Sigma + \neg\phi$ is inconsistent. An inconsistent set implies a contradiction; that is: -

(*) $\Sigma + \neg\phi \vdash \psi$ and $\Sigma + \neg\phi \vdash \neg\psi$ for some ψ

Hence $\Sigma + \neg\phi \vdash \psi \wedge \neg\psi$. From a contradiction, anything follows; hence, $\Sigma + \neg\phi \vdash \phi$. By the deduction theorem, $\Sigma \vdash \neg\phi \supset \phi$. By a tautology, $\Sigma \vdash \phi$, which contradicts the supposition.

The step that is not universal occurs at (*). It follows only from the assumption that $\Sigma + \neg\phi$ is *formally inconsistent*.

4.3 (+) Definition, formally inconsistent

A statement shall be said to be *formally inconsistent* if it implies by rules of deduction a formal contradiction. X is formally inconsistent if $X \vdash \phi \wedge \neg\phi$.

To say that $\Sigma + \neg\phi$ is *inconsistent* (the qualification "formally" being dropped) means: -

(**) $\Sigma + \neg\phi \vDash \psi$ and $\Sigma + \neg\phi \vDash \neg\psi$ for some ψ

A statement can be inconsistent without it being provable within a system of formal rules that it is inconsistent. For example, the statement "Arithmetic is inconsistent" if inconsistent could not be proven within arithmetic. In the proof, from (**) the conclusion $\Sigma \nvdash \phi \Rightarrow \Sigma + \neg\phi$

is consistent does not follow. So the proof has a "hint" of circularity, since the result represented by lemma [1] assumes the very thing that the whole theorem sets out to prove. The circularity can be *partially* avoided on the understanding that the concept of formal inconsistency follows from some background property that is possessed by Σ_0 but not by K. That is that every model of Σ_0 is a model in which every inconsistent statement can be formally proven to be inconsistent. So this implies that the model is a lattice in which the relation of deduction is compact. So the theorem should be recast in the form: *there exists a formal model of the first-order predicate calculus in which deductive proof paths are complete and compact:* $\Sigma \vDash \phi \Rightarrow \Sigma \vdash \phi$. The real task before us is not so much as to prove this theorem but to demonstrate the properties of the maximal model (lattice) wherein this property holds. Another form of this question is: what sets of statements can be added to the predicate calculus to produce theories that remain complete?

Lemma 2

$\Sigma + \neg\phi$ is consistent $\Rightarrow$ $\Sigma + \neg\phi$ has a model is consistent

To prove Lemma 2 we will develop a series of extensions to the theory Σ to obtain a theory Σ^* that has a model. Each theory is embedded in the next extension up; thus Σ is embedded in Σ^*, which means that any model of Σ^* is also a model of Σ_0. An important lemma that will be required to demonstrate this process is Lindenbaum's lemma.

Lindenbaum's lemma

If Σ is a consistent theory, then there exists an extension Σ^* of Σ such that, for any wff, either $\Sigma^* \vdash \phi$ or $\Sigma^* \vdash \neg\phi$. A consistent set that has this property is said to be *complete.*[7]

> Definition, rich / complete language (Monk [1976] slightly modified.)
>
> Let *L* be a first-order language. A set Σ is said to be *rich* if for every sentence of *L* that is of the form $(\exists x)\phi$ there is an individual constant *c* of *L* such that $\Sigma \vdash (\exists x)\phi(x) \supset \phi(c)$. Σ is said to be *complete* if, for every sentence ϕ of *L* either $\Sigma \vdash \phi$ or $\Sigma \vdash \neg\phi$.
>
> Monk's statement of Lindenbaum's lemma
>
> Any, complete, rich, consistent set of sentences has a model.

In the theorem the cardinality of the model is that of the set of all terms involved. For our purposes we assume the cardinality is $\aleph_0$.

[7] Crossley [1972] uses the term "rich". I am following Monk [1976], who uses "complete" for what Crossley calls "rich" and "rich" for something else.

Proof of Lindenbaum's lemma

Let $\phi_1, \phi_2, \dots, \phi_i, \dots$ be an enumeration of wffs of Σ. Define a sequence of theorems $\Sigma_1, \Sigma_2, \dots, \Sigma_i, \dots$ as follows

$\Sigma_1 = \Sigma$

$$\Sigma_{i+1} = \begin{cases} \Sigma_i + \phi_i \text{ if } \Sigma_\text{i} \nvdash \neg\phi_i \\ \Sigma_i \text{ otherwise} \end{cases}$$

Define $\Sigma^* = \bigcup \Sigma_i$. Then

(1) Σ^* is consistent.

Each Σ_{i+1} is an extension of Σ_i and Σ^* is the union of all such extensions. Hence if Σ^* is inconsistent it must be because one of the Σ_i is inconsistent. We prove by induction that all of the Σ_i must be consistent.

Firstly, $\Sigma_1 = \Sigma$ is consistent, by hypothesis.

Then, assume Σ_i is consistent.

If $\Sigma_{i+1} = \Sigma_i$ then Σ_{i+1} is consistent.

If $\Sigma_{i+1} \neq \Sigma_i$ then $\Sigma_{i+1} = \Sigma_i + \phi_i$ by definition. Then $\Sigma_i \nvdash \neg\phi_i$ and $\Sigma_i + \phi_i$ is consistent, by lemma [1]. Hence Σ_{i+1} is consistent.

(2) Σ^* is complete.

Let α be any closed wff of Σ.

Then $\alpha = \phi_{i+1}$ for some $i \geq 0$

Either $\Sigma_i \vdash \neg\phi_{i+1}$ or $\Sigma_{i+1} \vdash \phi_{i+1}$ for if $\Sigma_i \nvdash \neg\phi_{i+1}$ then ϕ_{i+1} is added to Σ_{i+1} as an additional axiom.

Hence, in Σ^* either $\Sigma^* \vdash \alpha$ or $\Sigma^* \vdash \neg\alpha$

Therefore, Σ^* is a complete, consistent extension of Σ.

4.4 Critique - concerning quantifier elimination

The criterion $\Sigma_i + \phi_i$ if $\Sigma_\text{i} \nvdash \neg\phi$ assumes a positive test for theoremhood, which is suspect. Mendelson [1970] p. 67 explicitly acknowledges this: to paraphrase what he says (slightly): "Note that even if one can effectively determine whether any wf is an axiom of Σ, it may not be possible to do the same with (or even to effectively enumerate) the axioms of Σ^*, i.e., Σ^* may not be axiomatic even if Σ is. This is due to the possibility of not being able to determine, at each step, whether or not $\neg\phi_{i+1}$ is provable in Σ_i." (Mendelson [1979] p. 67)[8]

There are two ways to deal with $\Sigma_{i+1} = \Sigma_i + \phi_i$ if $\Sigma_\text{i} \nvdash \neg\phi$: -

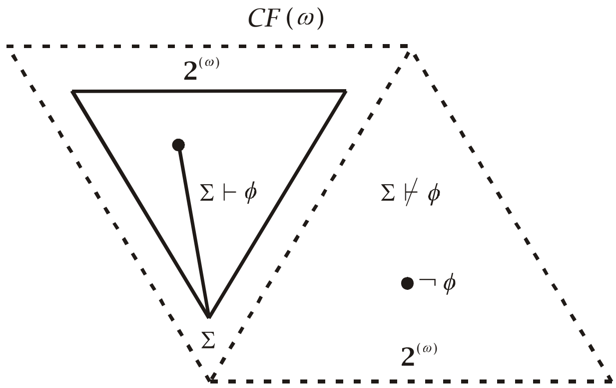
[8] The only modification is the change of the symbols so that they are consistent with the notation used here.

1. If each Σ_i is axiomatic then have a positive test for theoremhood and Σ^* is axiomatic. We can effectively determine each extension, decide whether to explicitly add $\neg\phi$ and so forth.
2. If any Σ_i is <u>not axiomatic</u>, then we can construct the complete extension of Σ, but as a <u>theoretical entity only</u>; it places it in the category of other completeness axioms (in the topological sense). We cannot enumerate the theorems of Σ^* - we can only say, theoretically, of any wff, either $\Sigma^* \vdash \phi$ or $\Sigma^* \vdash \neg\phi$; there is no effective positive or negative decision procedure.

Now this really indicates what an extraordinary theorem the completeness theorem is. For the main part our primary object of concern is the predicate calculus with a countably infinite language. However, the proof is also constructed in such a way that one can infer a *generalised completeness theorem* that would apply to every logic whatsoever. But what the implications of such a theory would be for the issue of effective computability would be very different. Now let us assume for the moment that at the stage Σ_i we have a portion of the predicate calculus that is equivalent to some finite Boolean algebra. Then, indeed $\Sigma_i \nvdash \neg\phi$ is decidable because we have either $\Sigma_i \vdash \phi$ or $\Sigma_i \vdash \neg\phi$. So every extension that is obtained in this way is axiomatic. In this way the completeness theorem is simply equivalent to the finite Boolean representation theorem. [Chap. 5 Sec. 6] Now this assumption will be justified even for the predicate calculus as a whole if we can prove quantifier elimination for that calculus. So the true heart of the theorem, so far as the predicate calculus is concerned, is quantifier elimination.

If we do not have quantifier elimination, so that there is some extension Σ_i that is not axiomatic, meaning we do not have a decision procedure that gives $\Sigma_i \vdash \phi$ or $\Sigma_i \vdash \neg\phi$, then the theorem may still prove that the theory is complete, *only we must be careful as to what completeness here means.* Completeness is the meta-property, $\Sigma \vDash \phi \Rightarrow \Sigma \vdash \phi$, so if we extend the notion of completeness to an extension Σ_i where we don't have $\Sigma_i \vdash \phi$ or $\Sigma_i \vdash \neg\phi$, but still adopt the rule $\Sigma_{i+1} = \Sigma_i + \phi_i$ if $\Sigma_i \nvdash \neg\phi$ then we are allowing the notion of proof path $\vdash$ to be stretched. Indeed, the concept may be so stretched so as to be virtually indistinguishable from the consequence relation $\vDash$, whereupon completeness becomes a mere definition. The most obvious way to stretch the deduction relation is to allow for proof paths that are actually infinite; this is called ordinal logic. But actually infinite proof paths may be of theoretical interest, but they are *not effectively computable.*

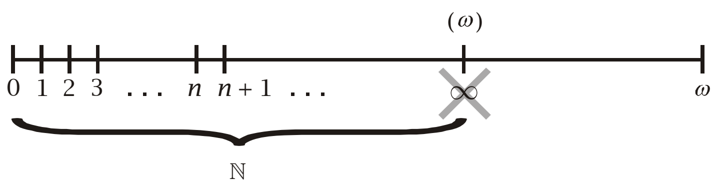
Thus, in conclusion, since our subject is effective computability it is correct to limit the proof path to compact proof paths, i.e. paths that are capable of being given a <u>finite</u> subsequence, even if they initally appear as infinite in length. When we eliminate quantifiers we specifically reduce a possibly infinite proof path to a finite one. To remain effective it is crucial that the criterion for each extension, $\Sigma_{i+1} = \Sigma_i + \phi_i$ if $\Sigma_i \nvdash \neg\phi$, be effectively decidable, hence that we have $\Sigma_i \vdash \phi$ or $\Sigma_i \vdash \neg\phi$ at every stage. This confines the models to those that possess the property of quantifier elimination.



We simply do not allow the proof path to be anything but compact, which means that there is a homomorphic image of it in embedded in some finite filter which is itself embedded in the ideal $\mathbf{2}^{<\omega}$ of $CF(\omega)$. Then, by the principle of dilution if $\Sigma_i \vDash \phi$ then we will finite prove $\Sigma_i \vdash \phi$; that places $\neg\phi$ in the ideal $\Sigma_i \nvdash \phi$. Thus, so far as the effectiveness of the predicate calculus is concerned, the predicate calculus is effective (1) if it has quantifier elimination and (2) its proof paths are homomorphic to finite proof paths in the lattice $CF(\omega)$.

4.5 Digression on the generalised completeness theorem

Some digression on the generalised completeness theorem is appropriate. Lindenbaum's lemma makes no reference to the size of the model; as far is it is concerned the model *could be non-denumerable.* However, the proof is by complete induction in the meta-language, which is in itself "evidence" in favour of Poincaré's thesis. (Here we will not assume this.) If we wished to extend this to a model of cardinality $> \aleph_0$ then we would have to use transfinite induction. That places us in a metalanguage that is some form of set theory. There is a proof in Mendelson [1979] (p.101) of the theorem due to Los and Rasiowa-Sikorski that the generalised completeness theorem is equivalent to the maximal ideal theorem; this means that the generalised completeness theorem depends on the Axiom of choice. (Or on the weaker Prime ideal theorem.) But this is to be expected. The diagram



indicates that when $\mathbb{N}$ is embedded in transfinite set theory, the role of ∞ in complete induction up to ∞ is replaced by the indeterminate boundary (ω) between the finite and the infinite; it is the role of the Axiom of Choice to assert that the element $\mu = \mathbb{N}$ (under different descriptions) is a set; from this we obtain the result known as the Maximal Ideal Theorem that $\mathbf{2}^{\mu}$ can be extended within $\mathbf{2}^{\omega}$ to a maximal ideal M. [5.7.21 et seq.]

Let us return to our annotated proof of the Gödel-Henkin completeness theorem.

The pre-extension[9]

1. Start with the theory Σ.

   In the framework of the completeness theorem as a whole, that starts with a theory $\Sigma + \neg\phi$ for some ϕ such that $\Sigma \nvdash \phi$. Now, without loss of generality, we will assume that our starting theory Σ just is this theory for which we have $\Sigma = \Sigma + \neg\phi$.

2. The aim is to provide a model for Σ, which is a first-order theory – that is, a theory embedded in the predicate calculus. To do this we must provide an interpretation for every possible statement of Σ, which includes quantifiers. We can start with either the existential quantifier as primitive and define the universal quantifier from it, or conversely. So we must show how we can validate any statement. Since we already know that the propositional calculus is complete, the problem to solve lies with the quantifiers. List all the formulae with v_1 as a free variable. That is $\psi_0(v_1), \psi_1(v_1), \ldots$[10]. There is no reason to suppose that the theory Σ decides any of these formulae. Indeed, they are open formulae, and not closed ones.

3. Add individual constants, $b_1, b_2, \ldots.$, to the language. These will be called "witnesses" and the new theory will be denoted Σ^0.

   Result

   Σ^0 is a consistent extension of Σ. (See Mendelson [1979] p. 68)

   It is the witnesses that will map directly to "objects" under the interpretation that will come later, and hence they provide the essential "glue" to stick the extended theory to the model.

4. Add new axioms of the form

   $(\exists v_1)(\psi(v_1)) \supset \psi(b_j)$

   for each appropriate witness.[11]

---

[9] This term is my own. It is a "pre-extension" that occurs prior to the application of the Lindenbaum lemma.

[10] Mendelson [1979] distinguishes the variables from each other. However, since v_1 is free there is no loss of generality in using it as the same dummy indeterminate variable in each formula.

[11] Mendelson [1979] takes the universal quantifier as primary and adds the axioms in the form: -
$\neg(\forall v_1)(\psi(v_1)) \supset \neg\psi(b_j)$.

4.6 Commentary

This line acts in the role of quantifier elimination which shall be discussed below. We have in this step eliminated the existential quantifier, replacing it by a concrete representative. I have already noted Monk's definition of rich above. The proposition that follows immediately confirms that the purpose of these axioms is to *eliminate quantifiers.*

4.7 Proposition (Monk [1976]

Let Σ be rich. Then for any sentence ϕ there is a quantifier free sentence ψ such that $\Sigma \vdash \phi \equiv \psi$.

4.8 On denumerable languages

There is an implicit assumption that the collection of witnesses $b_1, b_2, ...$ is denumerable. But there is nothing in the construction itself to guarantee this. In other words, this part of the proof would work with non-denumerable languages as would be the case in the generalised completeness theorem. Thus we identify the background property of the predicate calculus and complete theories therein: *that the predicate calculus is a denumerable language*; therefore, only a denumerable list of witnesses is required and the model is denumerable. If the language is not denumerable then the theorem will still go through, but the model will not be denumerable. It is only the finiteness of the language that guarantees it completeness. If the list of witnesses is not countably infinite, then the completeness theorem may still go through but cannot be used in a demonstration that whatever is written in such a theory is effectively computable. Any actual non-countable infinite collection is not effectively computable.

Continuation

There will be one new axiom for each formula listed at stage 2. The new theory, with all the additional axioms, which is an infinite list, will be denoted Σ^∞.

Result

Σ^∞ is a consistent extension of Σ^0. (See Mendelson [1979] p. 68)

Remark

There is some ambiguity as to the status of the collection Σ^∞ - is it potentially or actually infinite. If it were actually infinite and then quantified over, then the proof would become non-effective – at least as to *what the proof is about* – formalists would still insist that the proof itself was a finite object. However, my view is that Σ^∞ represents a potential infinity only.

Outline of the proof of lemma 2

1. Starting with Σ construct the pre-extensions Σ^0 and Σ^∞.
2. Then, with Σ^∞, apply Lindenbaum's lemma to obtain an extension Σ^* of Σ^∞ such that, for any wff, ϕ, either $\Sigma^* \vdash \phi$ or $\Sigma^* \vdash \neg\phi$.
3. Demonstrate the existence of a model for Σ^*.

Constructing the model

Define an interpretation *I* for Σ^∞ as follows.

A. The domain of *I* shall consist of all closed terms of *K*. (A closed term is a term without a variable).

B. The interpretation assigns each term to itself. That is *I* maps $t \to t$.

4.9 The crucial step: making the language a model of itself

This is the crucial step: it makes the language a model of itself. The relations in the language form a lattice. Since the lattice is countably infinite, it is equivalent to the atomless Boolean algebra. Recall that all countably infinite atomless Boolean algebras are isomorphic and that $CF(\omega)$ is its cannonical model.

C. *I* maps $f \to f^*$ where $f^*(t_1, t_2, ..., t_n) = f(t_1, t_2, ..., t_n)$

4.10 The language is a model of itself

The interpretation is just the language itself. So if we are mapping terms to terms, the referents of functions are just the wffs used to denote those functions. Hence the seeming vacuity of the definition here. The model is more or less a direct copy of the language – in every respect. So if *t* is a term then *t* is an individual in the model; if *f* is a wff representing a function, then the function just will be *f*, and so on. We denote the function f^* but we might (almost) have just used *f*. Note that functions map terms to terms.

D. *I* maps $R(t_1, t_2, ..., t_n) \to T$ iff $\Sigma^* \vdash R(t_1, t_2, ..., t_n)$.

Result

I provides a model *M* for *K*.

Proof

Since all theorems of *K* are theorems of Σ^{∞} and Σ^{∞} is a consistent extension of Σ^{∞}, it follows that any model of Σ^{∞} is also a model of Σ, hence we have to show only *I* provides a model *M* for Σ^{∞}.

Proof is by induction on the length of an arbitrary formula α where the length is defined to be the number of connectives and quantifiers in α.

(*i*) If α is a closed, atomic wff, then α is true in *M* iff $\Sigma^* \vdash \alpha$. This follows by the definition of *I*.

(*ii*) Induction step.

Induction hypothesis. If β is shorter than α then

β is true in *M* iff $\Sigma^* \vdash \beta$.

Proof by cases

We consider only the cases $\alpha = \neg\beta, \alpha = \beta \supset \gamma, \alpha = (\exists x)\beta$ since all other connectives and quantifiers can be defined in terms of these.[12]

Case (*a*): $\alpha = \neg\beta$

Suppose α is false, then β is true; hence, by the induction hypothesis $\Sigma^* \vdash \beta$. Since Σ^* is consistent, $\Sigma^* \nvdash \neg\beta$; that is, $\Sigma^* \nvdash \alpha$.

Suppose α is true, then β is false; hence by the induction hypothesis, $\Sigma^* \nvdash \beta$. By the completeness of Σ^* $\theta_{\max}$, $\Sigma^* \vdash \neg\beta$; that is, $\Sigma^* \vdash \alpha$.

Case (*b*): $\alpha = \beta \supset \gamma$

The proof is similar to that for (*a*).

Case (*c*): $\alpha = (\exists x)\beta$ [13]

[12] For a version of the proof based on the universal quantifier this line becomes:
$\alpha = \neg\beta, \alpha = \beta \supset \gamma, \alpha = (x)\beta$

[13] The proof that follows needs to be adapted to the universal quantifier as follows:-
Case (*c*): $\alpha = (x)\beta$: Let $\beta = F(\gamma)$. Assume $x = y$, for otherwise *x* is not free in β and in this case α is true in *M* if, and only if, β is true in *M*. By the induction hypothesis $\Sigma^* \vdash \beta$ iff β is true in *M*, and so for α as well. Hence $\alpha = (x)F(x)$. (*i*) Assume α is true in *M*, but $\Sigma^* \nvdash \alpha$. By completeness, $\Sigma^* \vdash \neg\alpha$; i.e. $\Sigma^* \vdash \neg(x)F(x)$. Hence $\Sigma^* \vdash \neg F(b)$ for some *b*. [This requires a separate lemma, see below.] But as $(x)F(x)$ is true in *M*, by the induction hypothesis $\Sigma^* \vdash F(b)$. That is $\Sigma^* \vdash F(b) \wedge \neg F(b)$. Hence, by contradiction $\Sigma^* \vdash \alpha$. (*ii*) Assume α is false in *M*, but $\Sigma^* \vdash \alpha$ Since $(x)F(x)$ is false in *M*, for some term t we have $F(t)$ is false in *M*. But $\Sigma^* \vdash (x)F(x)$, hence $\Sigma^* \vdash F(t)$, and by the induction hypothesis $F(t)$ is true in *M*. This contradicts the consistency of Σ^*, hence $\Sigma^* \nvdash \alpha$.

Let $\beta = F(\gamma)$. Assume $x = y$, for otherwise *x* is not free in β and in this case α is true in *M* if, and only if, β is true in *M*. By the induction hypothesis $\Sigma^* \vdash \beta$ iff β is true in *M*, and so for α as well.

Hence, let $\alpha = (\exists x)F(x)$.

(*i*) Assume α is true in *M*, but $\Sigma^* \nvdash \alpha$. By completeness, $\Sigma^* \vdash \neg\alpha$; i.e. $\Sigma^* \vdash \neg(\exists x)F(x)$. Hence $\Sigma^* \vdash \neg F(b)$ for some *b*. But as $(\exists x)F(x)$ is true in *M*, by the induction hypothesis $\Sigma^* \vdash F(b)$. That is $\Sigma^* \vdash F(b) \wedge \neg F(b)$. Hence, by contradiction $\Sigma^* \vdash \alpha$.

(*ii*) Assume α is false in *M*, but $\Sigma^* \vdash \alpha$ Since $(\exists x)F(x)$ is false in *M*, for some term t we have $F(t)$ is false in *M*. But $\Sigma^* \vdash (\exists x)F(x)$, hence $\Sigma^* \vdash F(t)$, and by the induction hypothesis $F(t)$ is true in *M*. This contradicts the consistency of Σ^*, hence $\Sigma^* \nvdash \alpha$.

This completes the proof that *M* is a model for Σ^* and hence for Σ, and thereby completes the proof of lemma (2).

This completes the proof that *M* is a model for Σ^* and hence for Σ, and thereby completes the proof of lemma (2).[14]

4.10 Observations on this non-constructive proof of completeness

This completes the description of the non-constructive proof of the completeness theorem. To summarise the observations made about this theorem.

1. As a theorem about first-order predicate calculus it is based on the fact that the language is countably infinite, and constructs a model for the

[14] However, in the version of the proof based on the universal quantifier this proof requires the additional following lemma.

Lemma (3)

$\vdash \neg(x)F(x) \rightarrow \vdash \neg F(b)$

The intuitive idea here is that if $\neg(x)F(x)$ is true then there must be some object for which *F* does not hold, and for the purposes of the proof we give it an arbitrary name '*b*'. The principle is clearly valid. In a full proof, however, in order to guarantee this, we have to first add to *K* a new set of symbols, $\{b_1, b_2, ..., b_i, ...\}$ such that no two are the same and then all axioms corresponding to the schema -

$\neg(x_i)F(x_i) \supset \neg F(b_j)$

This calls for an additional construction, and for a proof that the resulting theory is a consistent extension of *K*. In the existential version of the theorem, this has more or less already been done.

language itself out of the language. This model is a countably infinite lattice and hence must be the atomless countably infinite Boolean algebra. From this we see automatically that all proof paths in first-order predicate calculus must be compact; hence capable of finite representation. Proof in the predicate calculus therefore becomes effectively computable in the sense of a positive test of theoremhood.

2. It is the fact that the language is countably infinite that permits the essential step of quantifier elimination. Quantifier elimination means that all quantifiers in the predicate calculus are dummy symbols only. In terms of effective computing it means that any valid inference of the predicate caculus can be effectively simulated on a machine that uses only Boolean 2-valued functions. This is practically why the predicate calculus is effective.

3. The predicate calculus does not embed a principle of complete induction; however, complete induction is required in the meta-language to prove that the predicate calculus is complete. Therefore, the very proof that the predicate calculus is effective is premised on another principle that is, relative to the predicate calculus, non-logical. Certainly, the completeness proof is no proof of the effective computablity of complete induction – in makes no reference in the object language to it whatsoever but rather exploits its properties in the metalanguage.

An existential quantifier $(\exists x)\phi(x)$ just is a name of a lattice point – namely a disjunction of statements: $\phi(b_1) \vee \phi(b_2) \vee \ldots$. The disjunction could be finite or infinite, but the language, assuming it is denumerable, can only name a denumerable number of them. Therefore, regardless of what the underlying lattice is, and how many connections it has, and whether these are more than denumerable, the lattice to which the predicate calculus corresponds is a denumerable sub-lattice of this. Then every existential quantifier can be replaced by a single name of a lattice point. Here the device is used to select one of the particular members of the list, $\phi(b_1) \vee \phi(b_2) \vee \ldots$, as a *representative* of the list as a whole, or rather, the lattice point. So the background assumption is firstly that for any inference in a complete first-order theory a denumerable model is sufficient to represent it.

4.11 Constructive proofs of completeness

Since our interest is the theory of effective computablity the treatment here of the constructive proofs of the completeness theorem shall be light. The aim of a constructive proof is to provide an actual effectively computable procedure that starting with the question does Σ prove ϕ where both are wffs of the predicate caculus decides yes if indeed $\Sigma \vdash \phi$. It is a one-sided test, since it does not give $\Sigma \nvdash \phi$. The simple point is that every constructive method uses quantifier elimination at some point to reduce a problem involving quantifiers to one that does not, and then uses the decision procedure of the propositional calculus to finalise the problem. I will omit the method that uses Skolem functions as described by Wolf [2005] and Monk [1976]. Smulyan [1995] describes the method of analytic tableaux based on the Hintikka set. A very similar approach is offered by Boolos and Jeffrey [1980]. What they call a c*anonical derivation* is equivalent to a tableau.

Definition, canonical derivation (Boolos and Jeffrey [1980] p.131)

A *canonical derivation* from Δ is a derivation Γ such that

1. Every sentence in Δ occurs in Γ.
2. If $(\exists v)\phi$ occurs in Γ, then $\phi(t)$ for some term *t* occurs in Γ.
3. If $(\forall v)\phi$ occurs in Γ, then $\phi(t)$ for some term *t* occurs in Γ.
4. If $(\forall v)\phi$ occurs in Γ, then every possible substitution instance for $\phi(t)$ for some every possible term *t* occurs in Γ.
5. All function symbols appearing in Γ appear in Δ.

The whole point of a canonical derivation is that it tests for validity by eliminating quantifiers. This in turn constructively proves completeness because, "If Δ is unsatisfiable, any canonical derivation from Δ will be a refutation of Δ". (Boolos and Jeffrey [1980] p.131) Note that this is equivalent to completenss in our sense, $\Sigma \vDash \phi \Rightarrow \Sigma \vdash \phi$, because if $\Sigma \vDash \phi$ then $\Delta = \Sigma + \neg\phi$ is inconsistent, hence has no model (is unsatisfiable), hence, by the result, has a refutation. The refutation constitutes the proof path from Σ to ϕ.

4.12 There are no true quantifiers in complete predicate logic

We have already seen that the predicate calculus is not a true predicate calculus, since all predicates in it can be eliminated in favour of names of lattice points. Now I observe that if we have a predicate calculus such that *in the very notion of proof* all quantifiers can be eliminated, then *there are no true quantifiers in this logic* – and it is *not a quantifier logic either*. These surprising results may appear paradoxical, strange and doubtful, but they are true nonetheless. The root source of this result stems from the fundamental principle of formal analytic logic, namely to build a theory of logic over a lattice that is in turn constructed from a partition of space. The air of paradox comes from the fact that we are attempting to squeeze natural language into this straightjacket and certain residual intuitions continue to obstruct this movement. Indeed, it is the very thesis that is under examination here is that there are synthetic principles of reasoning, and no constraint upon those principles will ever make them fit into formal analytic logic. The tension here arises from the residual commitment to an alternative and unrecognised concept of inference – one that is based on meanings, not extensions. The issue is similar to the one that we observed when considering the paradox of material implication. [See Chap. 4 Sec. 6] If we wish to construct a true predicate calculus and a true logic of quantifiers we must start with something other than formal, analytic first-order predicate logic.

To return to the presentation by Boolos and Jeffrey [1980], their version and proof of the compactness theorem is instructive.

Compactness theorem

Δ is unsatisfiable iff some finite subset of Δ must be unsatisfiable.

Proof

The if part is trivial. For the only if part: Let Δ be unsatisfiable. Put Δ into prenex normal form. Then there is a canonical derivation Γ from Δ such that some finite set $\{\phi_1, \ldots, \phi_n\}$ of quantifier free sentences in Γ is unsatisfiable. Then delete from Γ all sentences that occur after ϕ_n. The result is a finite derivation from Δ that includes only a finite number of members of Δ. By the soundness theorem this finite subset of Δ is unsatisfiable.

4.13 The essentially finite character of the predicate calculus

The whole essence of the compactness theorem is that the proof of unsatisfiability must be finite, so only a finite subset of Δ can be involved in it. If we allow infinite paths then this result fails. The complete predicate calculus is *potentially infinite* but *actually finite*. That is to say, in any concrete instance of finding a proof of $\Sigma \vdash \phi$, given $\Sigma \vDash \phi$, the information encoded in $\Sigma + \phi$ is finite. So the completeness of the predicate calculus is really based on complete induction (in the non-set-theoretical sense), namely that if $\Sigma + \phi$ encodes information of finite degree *n*, then it is possible to solve a problem that encodes information of degree $n+1$.

We may now consider the Löwenheim-Skolem theorem.

4.14 Löwenheim- Skolem theorem

If Δ has a model, it has a model with an enumerable domain.

Proof

Let Δ be a set of sentences with a model. This means that it is satisfiable. Therefore, there is a canonical derivation Γ from Δ and by soundness this is not a refutation of Δ. Thus Γ is an enumerable, *OK* [synonym of locally compact] set of sentences that provides an interpretation that matches Δ. It is a model of Δ. [These assertions follow from their versions of the completeness theorem and three lemmas presented there. (Boolos and Jeffrey [1980] Chapter 12.)]

Suppose Δ has no predicate letters. Then the only non-logical symbols appearing in Δ are sentence letters, and in effect the domain is an enumerable set of propositions with valuations assigned by the interpretation *I* of the canonical derivation. (Δ is a tautology.) Thus suppose Δ contains predicate letters. Then there is at least one term appearing in Σ, which is the set of quantifier-free sentences of the canonical derivation. But everying in the domain of *I* is the denotation of some term appearing in Σ. There are at most enumerably many such terms; hence the domain of *I* is enumerable. Therefore, Δ has a model with an enumerable domain.

The formulation here of the Löwenheim-Skolem theorem is known as the *downward* L-S theorem because it takes a theory with an uncountable model and constructs a countable one. There is also the upward Löwenheim-Skolem theorem:

4.15 Upward Löwenheim- Skolem theorem

If Δ has a model with a countably infinite domain, then it has a model of every non-countable cardinality.

4,16 Observation

The completeness theorem is based on constructing a countably infinite domain that acts as a model of a set of sentences Δ. All the arguments here demonstrate that this model is isomorphic to the non-atomic countably infinite lattice $CF(\omega) \cong \mathbf{2}^{(\omega)} \cup \mathbf{2}^{(\omega)}$. The essential factor is the ideal $\mathbf{2}^{<\omega}$. If I now take the product of $\mathbf{2}^{<\omega}$ with any other lattice whatsoever, of any cardinality whatsoever, I will obtain another lattice. Let $L \cong \mathbf{2}^{<\omega} \times L_\alpha$ where $|L_\alpha| = \alpha > \aleph_0$, then *L* will be a model of Δ with cardinality α.

5 The rule of generalisation

5.1 The lattice of ideals

$F(\mathbb{N})$ is a potentially infinite set all of whose subsets are finite. However, there are other ideals that share this property – consider the ideal

$F(2) = \{0,2,4, \ldots ,\{0,2\},\{0,4\},\{2,4\}, \ldots ,\{0,2,4\}, \ldots\}$

which comprises the ideal of all finite subsets of even numbers in $\mathbb{N}$. It requires the Axiom of Choice to transform $(2) = \{0,2,4,6, \ldots\}$ into a well-defined, well-ordered set and extend it to a maximal ideal with corresponding maximal element: $(2) = \{0,2,4,6, \ldots\}$. [Compare to the maximal ideal theorem – Chap. 5 / 7.21] This maximal element belongs to the completion of $F(\mathbb{N})$ and not to $F(\mathbb{N})$ itself.

Consider a representation of the lattice that orders the atoms according to ascending primes is useful: -

$\alpha_1 = \{2\}$, $\alpha_2 = \{3\}$, $\alpha_3 = \{5\}$, ... , $\alpha_k = \{p^k\}$

where p^k is the k th prime. Relative to this ordering of the atoms in $\mathbb{N}_\infty$ we obtain ideals of $F(\mathbb{N})$ corresponding to maximal elements of $\mathbf{2}^\omega$ of the form: -

$(2) = \{0,2,4,6, ...\}$ $(3) = \{0,2,4,6, ...\}$ $(5) = \{0,5,10,15, ...\}$ ...

None of these sets belong to $F(\mathbb{N})$. They are correlatives of $\mathbb{N}$ in $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda$, and represent potentially infinite, unbounded collections. Hence they have actually infinite sets as counterparts also belonging to the lattice $\mathbf{2}^\omega$ and part of the boundary region of sets that are actually infinite but not co-finite. I denote these: -

$[2] = \{0,2,4,6, ...\}$ $[3] = \{0,2,4,6, ...\}$ $[5] = \{0,5,10,15, ...\}$ ...

Their enumerations are the same, but implicitly they are conceived as actually completed collections, whereas the sets in the previous collection are not. There are 1-point compactifications of sets $(2), (3), (5), ...$ and that these may be written: -

$[2] = (2) \cup \lambda_2$ $[3] = (3) \cup \lambda_3$ $[5] = (5) \cup \lambda_5$ ... $[p^k] = (p^k) \cup \lambda_{p^k}$ ...

where $\lambda_2 = \{(2)\}$ and in general $\lambda_{p^k} = \{(p^k)\}$. This has the same function as the 1-point compactification of $\mathbb{N}$: to complete the potential infinite and produce a well-ordered set of actually infinite members suitable for the partition of the interval. For that purpose the set must have a last member, and this is represented by λ_{p^k}. We have, for example,

$$[2] = (2) \cup \lambda_2 = \{0,2,4,6, ... , ... , \lambda_2\} = \{0,2,4,6, ... , ... ,(2)\} = \{0,2,4,6, ... , ... ,\{0,2,4,6, ...\}\} .$$

5.2 Non-compact proof paths

Now we turn our attention to the issue of what implications this structure of the lattice has for analytic logic and inference in general. Our formal analytic logic is based on the principle that a sound inference is a form of dilution of premises; this also means that in lattice terms we infer up the lattice and never down it or sideways. The consequences of a proposition $\overline{p}$ corresponding to lattice point p is everything that lies in the filter generated by p. This filter is written, $\overline{p} \vDash$. The compact proof paths are written $\overline{p} \vdash$. We see that the compact proof paths are limited to regions of the lattice that are isomorphic to the domain $F(\mathbb{N})$ - the ideal of all finite subsets of $\mathbb{N}$; any transcendence to the region lying above $F(\mathbb{N})$ belonging to the boundary requires an infinite, not compact, proof path. We must adjoin an element λ_k from the prime filter Λ. In our logic the set $\overline{(2)} = \overline{\{0,2,4,6, ...\}}$ is the name of a disjunction of atoms – it is a lattice point in its own right and represents the filter of all consequences of that disjunction. As a matter of direct intuition, we see that the set $\overline{[2]} = \overline{(2)} \cup \lambda_2$ lies in the filter $\overline{(2)} \vDash$; hence the expression, $\overline{(2)} \vDash \overline{[2]}$, is a valid semantic consequence. However, that does not mean that we may automatically write $\overline{(2)} \vdash \overline{[2]}$ because *in practical cases we actually have to construct a proof*, and since the proof path is not compact we need some other device. Taking $\overline{(2)} = \overline{\{0,2,4,6,...\}}$ as a concrete example, this name of a lattice point represents some kind of proposition $P(k)$ that may be regarded as contingently given. It corresponds to a lattice point and generates a filter. In formal logic we have the rule of inference, Existential instantiation: -

$$\frac{P(k)}{(\exists n)P(n)}$$

However, if the logic has quantifier elimination then we are confined to a compact part of the lattice, that is, to a part isomorphic to $CF(\omega)$ and cannot transcend that part by means of this device, unless there is some other implicit principle involved. In fact, in a compact domain $P(k)$ and $(\exists n)P(n)$ are names of the same lattice point; $(\exists n)P(n)$ represents a join of lattice points, $(\exists n)P(n) \equiv P(a) \vee P(b) \vee ...$ and $P(k)$ is some arbitrary member of that join used to represent it.

5.3 Generalisation

The inference of Generalisation is more problematic: -

$$\frac{P(k)}{(\forall n)P(n)}$$

As indicated earlier [5/1.3] we could only have $\vdash (\forall n)P(n)$ on an unrestricted domain if in fact $(\forall n)P(n) \equiv \mathbf{1}$. Even on a restricted domain $(\forall n)P(n)$ stands for a meet of lattice points, and the inference, $P(k) \vdash (\forall n)P(n)$ is either a swapping of two names of the same lattice point, or perhaps an inference *down the lattice*, which would appear to be logically unsound. If it is an inference up the lattice then $(\forall n)P(n)$ is a *dilution* of the information contained in $P(k)$, so some additional information has been joined to it: $(\forall n)P(n) \equiv P(k) \vee \phi$.

5.4 Embedding complete induction in the lattice

(ω) is the indeterminate size (measure) of $\mathbb{N}$ in $\mathbb{N}_\infty$, which is the partition of $[0,1]$ into ω parts. It is not possible to determine precisely where the neighbourhood of 0 ends and the neighbourhood of 1 begins. In fact, they interpenetrate each other.

It is the Axiom of Choice that asserts that μ is also a set. This accounts for the relative strength of the Axiom of Choice (Zorn's Lemma) and the Boolean Prime Ideal theorem. The Axiom of Choice asserts that μ is a set; the Boolean Prime Ideal theorem asserts that $\mathrm{M} = \mathbf{2}^{\mu}$ is a set and the former implies the latter, but not conversely.

If we break up $[0,1]$ into ω actual atoms, we obtain a situation illustrated by the following diagram: -

This illustrates how the inclusion $\lambda = \{\mathbb{N}\}$ as an atom in the partition exactly parallels the inclusion of the other atoms. But the partition is a partition into an antichain, so the collection $\{\{0\},\{1\},\{2\},...\}$ *has no order structure upon it*, for otherwise the structure that is generated over it as skeleton would also be a chain and *not a Boolean (distributive) lattice.*

When the logic is sufficiently strong [Chap.9 / 1.6], then it is possible to embed a principle of complete induction and definition by recursion into the analytic structure. But where in the structure does this occur? It cannot appear in μ for that would then transform the collection $\{\{0\},\{1\},\{2\},...\}$ into a chain. Hence, the entire set $\mathbb{N}$ is embedded into the structure as a completely ordered chain on which complete induction is defined. $\lambda = \{\mathbb{N}\}$ is the atom that asserts that complete induction for the actual infinite is permitted. There are principles of relativity and symmetry at work here as well. From the perspective of 0 λ represents the neighbourhood of 1, and is a set comprising a completely well-ordered chain of co-atoms; 0 "perceives" λ as having a last element, this being the atom next to 1 in the actually infinite ascent from $\{0\}$ where the atoms $\{0\},\{1\},\{2\},...$ are placed in one-one correspondence with the *ordinals*, 0, 1, 2, ...; hence, complete induction by descent is defined on the interior of the atom $\lambda = \{\mathbb{N}\}$. For complete induction by descent to work, *starting with the last ordinal next to* ω we must be able to well-order λ; that is, λ must have a first-element. It requires the Axiom of Choice in order to prove that λ is well-ordered; this transforms λ into a set. Prior to the application of the Axiom of Choice λ was not even a set, let alone a well-ordered one. It was an ordered *collection* (as opposed to set) of co-atoms of the neighbourhood of 1 that has no least member; just as $\{0\},\{1\},\{2\},\ldots$ is an *unordered collection* of atoms of the neighbourhood of 0 that has no greatest element. The Axiom of Choice enables one to order this latter collection, provide it with a greatest member, and thus transform it from a mere collection into a set. Without the Axiom of Choice, the co-atoms in the neighbourhood of 1 are an unordered collection that can be represented as $\mathbb{N}_\infty-\{0\},\mathbb{N}_\infty-\{1\},\mathbb{N}_\infty-\{2\},\ldots$; under the Axiom of Choice these are well-ordered; we represent the co-atoms as $\{0\}',\{1\}',\{2\}',\ldots$; the Axiom of Choice places these into one-one correspondence with the ordered set of natural numbers equipped with complete induction. From the viewpoint of 1 the perspective is reversed; by symmetry the interpretation must be the same. The neighbourhood of 1 becomes the unordered collection of co-atoms: $\left\{\{0\}',\{1\}',\{2\}',\ldots\right\}\cong\mathbb{N}$ that serves as the analytic partition of this neighbourhood; the complements of these co-atoms, the co-co-atoms are equated with the atoms of the neighbourhood of 0; this collection: -

$\{\text{collection } \{0\},\{1\},\{2\},\ldots \text{ as chain}\}\cong\{\{0,1,2,\ldots\}\}\cong\{\mathbb{N}\}$.

is perceived by 1 as a single well-ordered set comprising a single atom that is the neighbourhood of 0. In summary: -

5.5 (+) Principle of symmetry

From 0 $\mu=\{0\}\cup\{1\}\cup\{2\}\cup\ldots=\text{antichain: }0,1,2,\ldots=\{0,1,2,\ldots\}\cong\mathbb{N}$.

$\lambda=\left\{\text{chain:}\{0\}',\{1\}',\{2\}',\ldots\right\}\cong\{\mathbb{N}\}$ as a complete well-ordered chain, where $\{n+1\}'=s\left(\{n\}'\right)=\{s(n)\}'$. (Here s represents successor.)

From 1 $\Lambda=\text{antichain:}\{0\}',\{1\}',\{2\}',\ldots=\{0\}'\cup\{1\}'\cup\{2\}'=\{0',1',2',\ldots\}\cong\mathbb{N}$

$\mathrm{M}=\{\text{chain:}\{0\},\{1\},\{2\},\ldots\}\cong\{\mathbb{N}\}$.

If 0 is a set, then it has a complement. Likewise, for 1, 2, 3, ... Hence the collection $\{0',1',2',\ldots\}$ is well-defined.

5.5 Representing induction in the analytic lattice

Consider the particular example: -

$$(\forall n)\left(1+2+\ldots+n\equiv\frac{1}{2}n(n+1)\right)$$

We see that this takes the form, $(\forall n)(f(n)\equiv g(n))$ where f and g are functions. Furthermore, these reduce to recursive relations: -

$$F(n,x) = \{(n,x) : x = 1+2+...+n\} = \{(1,1),(2,3),(3,6),\,...\}$$
$$G(n,x) = \left\{(n,x) : x = \frac{1}{2}n(n+1)\right\} = \{(1,1),(2,3),(3,6),\,...\}$$

The problem then reduces to showing that these two relations are extensionally equivalent; that is: -

$$F(n,x) = G(n,x)$$
$$\Leftrightarrow \quad \{(1,1)=(1,1),(2,3)=(2,3),(3,6)=(3,6),\,...\}$$
$$\Leftrightarrow \quad D_{F=G} = \{(1,1),(3,3),(6,6),\,...\} \text{ where } D_{F=G} \subset D_{x=y} = \{(0,0),(1,1),(2,2),\,...\}$$

If we interpret the pair (x,y) to represent an atom of the lattice, then all these expressions become disjunctive sets – joins of atoms in the lattice somewhere. Substitution into the formula $1+2+...+n \equiv \frac{1}{2}n(n+1)$ will establish particular members of the join $(1,1)\vee(3,3)\vee(6,6)\vee...$ but any finite iteration of this process will not enable one to infer that the statement is true of *all* members of this potentially infinite join. Representing the actually finite but potentially infinite join of any member of this sequence by $(D_{F=G})$ we still need to join something more to this in order to obtain the actually infinite collection $[D_{F=G}]$. The inference : -

$$(D_{F=G}) \vdash [D_{F=G}]$$

is an inference up the lattice and a sound one. Certainly, $(D_{F=G}) \vDash [D_{F=G}]$ is true as $[D_{F=G}]$ lies in the filter generated by $(D_{F=G})$, but we still have to establish the inference as a concrete proof. In fact, we only ever know a statement of the form $(\phi) \vDash [\phi]$ because we have a concrete proof *somewhere* that gives us $(\phi) \vdash [\phi]$. That proof might take place in the logic built directly over the lattice, or it might take place in some meta-language about the lattice; it might also take place in some logic that cannot be represented by a lattice whatsoever – the very possibility that formalists deny.

Now consider the form of complete induction in general: -

$$\begin{array}{l} P(1) \\ \underline{P(k) \to P(k+1)} \\ (\forall n)Pn \end{array}$$

The proposition $P(1)$ represents some lattice point, and defines a filter in it: $P(1)\vDash$. But what lattice point does the proposition $P(k) \to P(k+1)$ represent in the lattice? It says, *if* $P(k)$ defines a filter *then* $P(k+1)$ lies in that filter. This is not a contingent proposition, but a necessary one based on some argument concerning the order of the natural numbers: 0, 1, 2, ... k, $k+1$, ..., which has no direct representation in the lattice of all finite subsets, M. Suppose it did correspond to a lattice point in M, then it would be equivalent to the contingent join of some atoms of that lattice. So what is necessary would be contingent.

Furthermore, such a join would still be an element of M and by joining it to $P(1)$ we would still not transcend M. Thus, it is clear that the statement $P(k) \rightarrow P(k+1)$ does not correspond to any lattice point in M, and hence, if it is represented at all, it must be a lattice point belong to the filter of λ. The atom λ embeds into the lattice the whole <u>chain</u> $\mathbb{N}$ together with its synthetic principle of complete induction defined upon it, and thus permits in principle the inference $\overline{(\phi)} + \lambda_\phi \vdash [\overline{\phi}]$; here λ_ϕ represents complete induction on the set (ϕ) by mapping the well-ordering of (ϕ) to the succession of natural numbers contained in λ_ϕ. In general, the 1-point compactification: - $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\} = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda$ adjoins to the atoms $\mathbb{N}$ the principle of complete induction on λ by mapping that set of atoms to the natural numbers contained in $\mathbb{N}$.

5.6 The rule of generalisation

Let us again examine the rule of Generalisation: -

$$\frac{P(k)}{(\forall n)P(n)}$$

For this inference to be sound $(\forall n)P(n)$ must lie in the filter defined by $P(k)$, so represents also a dilution. In the compact region *it is nothing* but a switch between names of the same lattice point, since $P(k)$ is simply a name of the same lattice point as $(\forall n)P(n)$. When it is a device for inferring from the compact region *into* the boundary, it becomes a substantial principle, but just for that reason it *must be justified.* In practice, we never know that $(\forall n)P(n)$ lies in the filter $P(k)$ *unless there is a proof that this is so.* So while we have $P(k) \vDash (\forall n)P(n)$, we only know this if we have $P(k) \vdash (\forall n)P(n)$ and for that *we need a special argument that this is so.* This means that where the inference $P(k) \vdash (\forall n)P(n)$ transcends from the compact region to the boundary *it is a disguised form of mathematical induction.* A consideration of the simplest proofs of mathematics will demonstrate this. We infer (subject to Euclidean geometry) that, given an arbitrary triangle, Δ, that the angle sum of *every Euclidean triangle whatsoever* is 180º. This *appears* to be a classic instance of the inference: -

$$\frac{P(\Delta)}{(\forall \Delta)P(\Delta)}$$

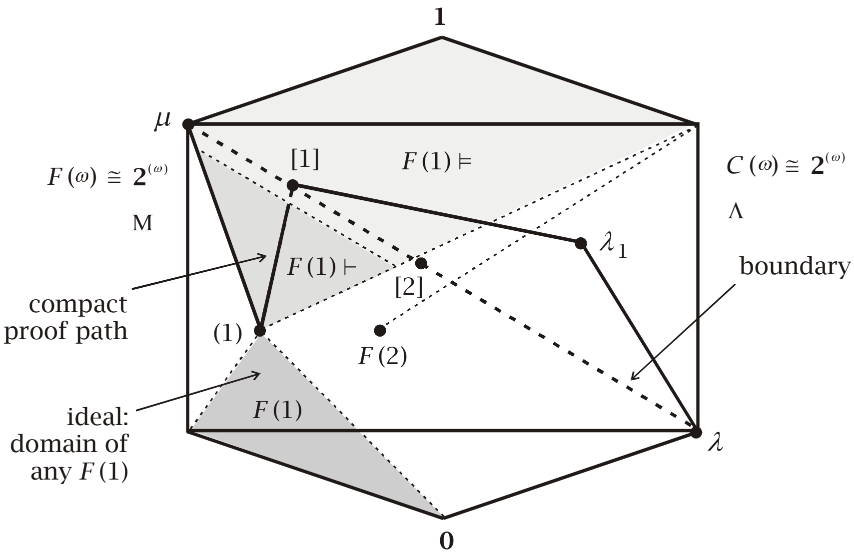
where Δ = arbitrary triangle. But how do we know that an arbitrary triangle is an arbitrary triangle? – That whatever applies to this arbitrary triangle applies to all triangles whatsoever? There is a disguised implicit induction in the argument that goes something like this: a continuous deformation of one triangle into another changes none of its essential properties.

For that to work the class of all triangles must be well-ordered and the continuous deformation is an induction argument on this underlying well-ordering. But this is an illustration only; what it shows is that where we have the inference $P(k) \vdash (\forall n)P(n)$ and $P(k)$ is not simply just another name of what $(\forall n)P(n)$ denotes, then there must be some additional justification for it.

5.7 The inference from any to all

In the lattice *any* corresponds to a proposition ϕx, which, as Whitehead and Russell [1910] explain, is an ambiguous and indeterminate symbol. Any yet we can operate with it *as if it were determinate* because what we are doing in our minds is substituting an arbitrary determinate value for *x* and working with that – that is ϕa for some "individual" *a*. The individual *a* is *any* member of the collection of all *x* that satisfy ϕ; hence is any member of the ideal of finite sets of which it is a representative. But $(x)\phi x$ correspond to the set of all *x* that satisfy ϕ so it is the closed and bounded totality; in an infinite lattice it is an actually infinite set. Therefore, in the lattice, the two points *any* and *all* are not identical.

Ideals		Infinite sets in the lattice	
(1)	any odd number	[1]	all odd numbers
(2)	any even number	[2]	all even numbers
M	any finite number	λ	any cofinite set



We have the relationships: -

$[1] = (1) \cup \lambda_1 \qquad [2] = (2) \cup \lambda_2 \qquad \ldots$

5.8 The inference all to any

On the principle that all sound inference proceeds up the lattice, the inference from *any* to *all* is sound; but the inference from *all* to *any* proceeds down the lattice, and hence is appears possibly unsound. All to any goes down the lattice; this is because, given any to all, it implies that the domain of arguments is well-ordered, so to progress all to any requires the axiom of choice. How do I know that an *arbitrary object* is representative of a whole class of objects? I pick an object at random – what guarantees that another object has the same *essential properties*? There is implied in this a rule that *the objects can be ordered* and *in progression from one object to another* no essence is lost.

CHAPTER 8

Generic sets

1 Generic sets

Generic sets were introduced by Cohen [1966] in order to prove (1) that there is a model of set theory (ZF) in which there are non-constructible sets, and (2) that the Axiom of Choice is independent of the axioms of ZF set theory. They were introduced in the context of a minimum transitive model of set theory.

1.1 Definition, transitive set

A set is said to be *transitive* if every one of its members is a subset.

<u>Example</u>

Ordinals are transitive sets. For example, $2 = \{0,1\}$ is transitive.

1.2 Definition, transitive model

Let A be a transitive set. If $\langle A, \in \rangle$ is a model of ZF, then it is said to be a *transitive model* of ZF.

1.3 Theorem

Let ZF have a transitive model. Then the intersection of all transitive models of ZF is a transitive model of ZFL. Furthermore, this transitive model takes the form L_λ, where λ is a denumerable limit ordinal, and hence also denumerable.

<u>Notation</u>

The model referred to in the above theorem is called the *minimum model* and is denoted, $\mathfrak{M}$.

The concept of a minimal model is problematic. I note the following points made about it from Wolf [2005]: (1) $\langle L, \in \rangle$ is not a true model of ZFL because L is a proper class, and the completeness theorem does not apply to classes. (2) L is said to be an *inner model* of ZFL. This means that L is a transitive proper class in which all the axioms of ZFL hold. (3) The completeness theorem says that a first-order theory has a model if and only if it's consistent. Thus, if ZF could prove that ZFL has a model, then ZF could also prove that ZFL is consistent, which would mean (by Gödel's second incompleteness theorem) that ZF is inconsistent. The minimum model has some relation to a minimal element of an filter in a lattice. I wish to discuss Cohen's concept of a *generic set* but to dispense with the difficulties raised by this problematic notion of a minimum model. In any event the Boolean and Stone representation theorems (they are different expressions of the same underlying "fact", subject to the Axiom of Choice) assert such a close relationship between set theory and lattices that one can take the lattice to the be the primary object, and the minimum model is a prime filter "defined" by a generic sequence of lattice extensions. (See also Chapter 11 for a discussion of definability.)

I shall start with a lattice *L*, which, like the minimum model $\mathfrak{M}$, has been constructed at some definite ordinal level where $\alpha_0 = \sup\{\alpha : \alpha \in L\}$. Any lattice point (corresponding to a set) existing in *L* is therefore constructed at some level below $\alpha \le \alpha_0$. The lattice points of *L* are represented by sets of numbers. For example $\{0\}, \{0,1\}, \{0,2,3\}$ are lattice points. Each lattice point defines a filter. For example, the filter $0 \in x$ corresponds to the filter which contains every set in which 0 appears: $0 \in x = \{\{0\},\{0,1\},\{0,2\}, \ldots ,\{0,1,2\},\{0,1,3\}, \ldots\}$. This can also be denoted, $\text{filter}\{0\}$.

The model shall be countably infinite; hence *it cannot be atomic*. [5/4.2] This makes it into a model of the algebra of statement bundles etc. There are no atoms; however, relative to a numbering of the lattice, there is a *notional floor* to the lattice and this gives *notional atoms* [5/5.8]; this floor can be lowered, so that below any level there is another level. Nonetheless, since the model is countably infinite it is built over a partition that can be numbered by $\mathbb{N} = \{0,1,2,\ldots\}$. Thus, although *L* is not atomic there is some level in the lattice corresponding the singleton sets $\{0\}, \{1\}, \{2\}, \ldots$; these singleton sets contain individuals, which are natural numbers $m \in \mathbb{N}$. It is these singleton sets that correspond to *notional atoms*. To each individual $m \in \mathbb{N}$ there corresponds a filter *x*. Thus each individual corresponds to a filter, $m \in x$. For example, $0 \in x$ and $1 \in x$ denote the filters corresponding to the lattice points (notional atoms) $\{0\}$ and $\{1\}$ which are their infima.

Example

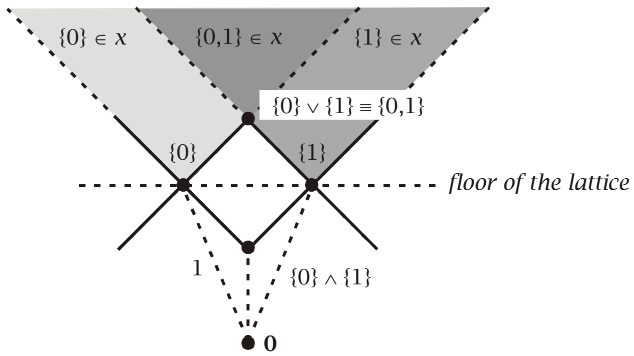
$0 \in x$ corresponds to the filter which contains every set in which 0 appears:

$$0 \in x = \{\{0\},\{0,1\},\{0,2\}, \ldots ,\{0,1,2\},\{0,1,3\}, \ldots\}$$

To further illustrate...

$$\begin{aligned}\{0,1\} &\leftrightarrow \{0,1\} \in x \\ &= (0 \in x) \vee (1 \in x) \\ &= \{\{0,1\},\{0,1,2\},\{0,1,3\}, \ldots ,\{0,1,2,3\},\{0,1,2,4\}, \ldots\}\end{aligned}$$

Because the singleton sets $\{0\}, \{1\}, \{2\}, \ldots$ do not represent *true atoms*, their meets define filters. However, *we cannot represent these filters by sets without lowering the floor of the lattice.* [5/5.8] Lowering the floor of the lattice means rewriting the lattice so that what was a singleton set becomes a set with more than one element and new singleton sets are introduced representing the lower floor. This is also equivalent to splitting the individuals; lowering the floor of a lattice is similar to splitting the atom in physics – what was originally conceived as incapable of further division is revealed to be a combination of more primitive individuals. Another way of putting this is to say that we embed the original lattice in a new larger lattice and the embedding maps singleton sets in the former to non-singleton sets in the latter. Given a non-atomic lattice with singleton sets, we cannot represent their meets by other singleton sets, but only by expressions of the form $(0 \in x) \wedge (1 \in x)$ and so forth.



In the metric of the lattice L the lattice point $\{0\} \wedge \{1\}$ lies between **0** and the floor, so strictly cannot appear as a lattice point of L. Nonetheless, the filters $(0 \in x)$ and $(1 \in x)$ are defined and so is their meet $(0 \in x) \wedge (1 \in x)$. So by conjunctions of filters we *lower the floor* of the lattice. To represent meets such as $(0 \in x) \wedge (1 \in x)$ as sets, we must *split the notional atoms*, that is, increase the partition, which is what lowering the floor means.

> **Example**
> We may replace the notional atom $\{0\}, \{1\}$ and $\{2\}$ by infinite, incomplete sets of atoms, say: -
> $\{0\} \equiv \{a,b,c,d,...\}$ $\{1\} \equiv \{a,b,d,...\}$ $\{2\} \equiv \{a,c,d,...\}$
> $\{0\} \wedge \{1\} \equiv \{a,b,d...\}$
> $\{0\} \wedge \{1\} \wedge \{2\} \equiv \{a,d,...\}$
> Alternatively, we can switch to an interval algebra. [6/2.7]

If the lattice is countably infinite to begin with, then lowering its floor (splitting its notional atoms) does not take the lattice extension out of the class of countably infinite lattices. The individuals of a lattice resulting from such a *finite* or *potentially infinite* process may always be regarded as *countable numbers*; this being a consequence of the property of $\mathbb{N}$ that doubling the members of $\mathbb{N}$ does not result in a set of larger cardinality: $2\mathbb{N}$ is equinumerous to $\mathbb{N}$. To obtain a larger set one must take a limit – that is here, extending the process of lowering the floor to actually infinite iterations. The result is a lattice of greater cardinality than $\mathbb{N}$, one that encompasses generic sets.

1.4 Lowering the floor of a lattice

The discussion of generic sets is usually based on a distinction between a language, here *K*, and a model *M*. The construction can be defined solely in terms of the model, which here is replaced by a lattice *L*, on the principle that every model is a lattice. I propose to show how generic sets may be constructed for any of these. The specific lattice in question shall be the non-atomic lattice of countably infinite lattice points, denoted by L_0, which is also the domain of the effectively computable [Chap.7, Sec.3]. Though non-atomic it shall be assumed that relative to some language this lattice has a notional floor – that is, a collection of singleton sets representing notional atoms. Because it is not atomic, it is possible to lower the floor and obtain lattice points below it. This is represented by an embedding of the lattice L_0 in a larger lattice L_1. Iteration of this process creates a sequence of lattice extensions $L_0, L_1, \dots, L_k, L_{k+1}, \dots$. We use the language *K* to discuss the relations between any lattice L_k in this sequence and its extension L_{k+1}.

The language, *K,* is built over the lattice L_0, with given individuals *m* and hence singleton sets (notional atoms) $\{m\}$. To the lattice filter $m \in x$ there corresponds the statement in *K*, $\overline{m} \in \overline{x}$, where $\overline{m}$ and $\overline{x}$ are said to be *labels* or *names* of *m* and *x* respectively. The expression $\overline{m} \in \overline{x}$ is a *statement* of the language *K.*

Example

$0 \in x = \{\{0\},\{0,1\},\{0,2\}, \dots ,\{0,1,2\},\{0,1,3\}, \dots\}$ defines a filter in a lattice. The corresponding statement in the language is $\overline{0} \in \overline{x}$.

Thus the language closely matches the lattice, but the language has other statements and symbols that enable it to *talk about* the lattice and so discuss its extensions and embeddings in larger lattices. We introduce the symbol $\Vdash$ to denote a relation called *forcing* between statements of *K* to describe relations between filters in L_0. Whenever the lattice point q is contained in the filter defined by the lattice point *p* and the two are connected by a compact proof path [6/2.3], this is denoted by $\overline{p} \vdash \overline{q}$.

Example

Let $\overline{p} \equiv \overline{0} \in \overline{x}$ and $\overline{q} \equiv \overline{\{0,1\}} \in \overline{x}$. Then $\overline{p} \vdash \overline{q}$.

1.5 Forcing

The relation of forcing is such that

$\overline{p} \vdash \overline{q} \quad \Rightarrow \quad \overline{p} \Vdash \overline{q}$

but the forcing relation extends the idea of deductive consequence of statements $\overline{p}$ beyond that of deductive inference or its lattice equivalent *compact proof path*. The relation of forcing is thus connected to that of logical consequence. If a lattice is not complete then we have some lattice points such that $\overline{p} \vDash \overline{q}$ but $\overline{p} \nvdash \overline{q}$. The distinction between forcing and consequence is as follows. When we consider consequence, where we have incompleteness, we have a prior conception of the lattice *L* that has non-compact proof paths. By contrast, the forcing language is built over a lattice *L* that is complete in this sense, but enables one to discuss the relation of that complete lattice to other lattices that are generic extensions of it. So forcing is a relation between lattices whereas consequence is a relation within a given lattice. We shall see that the forcing relation demonstrates that any countably infinite, non-atomic lattice must be embedded in another lattice which is complete as a lattice but contains non-compact proof paths, hence logically incomplete [See also Chap. 9]. The "must" in this last sentence is highly significant. All analytic relations are contained within a prior conceived lattice, and any member of the sequence of lattice extensions $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ obtained by finite iterations of the generic construction also corresponds to an analytic logic built over it. But the lattice that arises as the actual limit of this sequence, $L_\omega = \lim_{n\to\infty} L_n$, is not the analytic continuation of any of these: it is not derived from the analytic logic of any of these. The process of constructing a generic set, and hence a lattice that is the limit of a sequence of generic extensions, is a *synthetic relation of necessary truth*. The forcing notion is not a property of formal, analytic logic and is not effectively computable.

We need to demonstrate the general nature of a generic sequence. We have a lattice L_0 and a language *K* built over this lattice and equipped with a relation of forcing $\Vdash$ subject to the rule $\text{-}\overline{p} \vdash \overline{q} \Rightarrow \overline{p} \Vdash \overline{q}$ where $\overline{p} \vdash \overline{q}$ iff there is a compact proof path in L_0. The language *K* is countably infinite and hence all statements of *K* can be recursively enumerated; this corresponds to a recursive enumeration of the lattice points of L_0. Let $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$ be any recursive enumeration of the statements of *K*. These are statements of the form $\overline{m} \in \overline{x}$ where *m* is a lattice point and *x* is a filter of L_0.

1.6 Sets, filters and labels

The relation between sets and filters can cause difficulties, so it is useful to digress to clarify what is happening here. This is because of the nature of the subset / supraset relation. In a countably infinite lattice L_0 with notional singleton sets representing notional atoms, the singleton (atom) $\{0\}$ defines a filter of which $\{0\}$ is always a subset of every element.

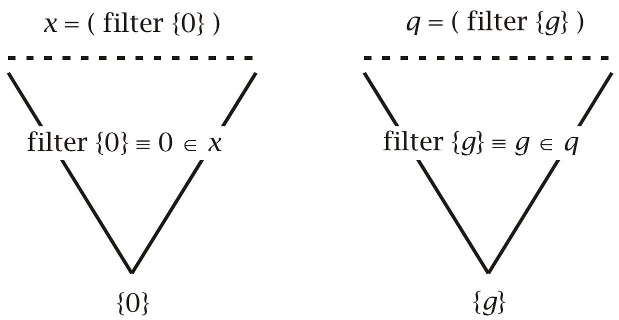
This filter, denoted *x*, is constructed as follows: -

$x = \text{filter}\{0\} = \{\{0\},\{0,1\},\{0,2\}, \dots ,\{0,1,2\},\{0,1,3\}, \dots\}$

Denoting the elements of the filter by *y*, we have: -

$(\forall y)(y \in x \text{ iff } 0 \in y \text{ iff } \{0\} \subseteq y)$

So an equivalent representation of the filter is $0 \in x$, and the corresponding label in *K* is $\overline{0} \in \overline{x}$. This presents *x* as a large set of which every element of the filter is a supraset of $\{0\}$: The set $\{0\}$ represents the minimum lattice point of the filter and is such that for all $y \subseteq x$ we have $\{0\} \subseteq y$. Likewise, 0 represents an individual that is the notionally unique member of $\{0\}$. When we extend this to the construction of a generic set, we obtain *q* as the generic set, but in the expression $m \in q$ also represents the filter. To do so we must assume the Boolean prime ideal theorem, which allows any sequence of filters to be extended to a maximal (prime) filter. Here that maximal filter (ultrafilter) is denoted *q*.



We also have a new notional atom (singleton) $\{g\}$ such that $\{g\} \subseteq q$, and this creates a new notional individual *g*. The set $\{g\}$ has a claim to be the "generic set", though following common practice we reserve this designation for *q*, which is the $\text{filter}\{g\}$.

1.7 Rules for the construction of generic sets

We now establish a series of rules for the generic construction which leads to a series of lattice extensions $L_0, L_1, \dots , L_k, L_{k+1}, \dots$ and the definition of a *generic set*. Let this generic set be denoted by *a* in a lattice and let $\overline{a}$ be label in *K*. Recall that $\overline{S}_1, \overline{S}_2, \dots ,\overline{S}_k, \dots$ is a recursive enumeration of all statements of *K*.

<u>Rule 0</u>

We start with a consistent set $\overline{p_0}$ of statements.

From the outset it is useful to recognise that the set-theoretic representation of the set of finite conditions $\overline{p_0}$ is either confusing or an actual error. The set $\overline{p_0}$ represents some *finite information* about a generic set *q*, and comprises a finite list of *conjunctions* of the form $\overline{m} \in \overline{q}$ or $\overline{m} \notin \overline{q}$. For example: -

$\overline{p_0} = \left\{ \overline{3} \in \overline{q},\ \overline{47} \notin \overline{q},\ \overline{932} \in \overline{q} \right\}$

But this customary use[1] of curly brackets here is misleading, since $\overline{p_0}$ is a conjunction of conditions, not a disjunction of them. That is, we should write: -

$\overline{p_0} = \left(\overline{3} \in \overline{q} \right) \wedge \left(\overline{47} \notin \overline{q} \right) \wedge \left(\overline{932} \in \overline{q} \right)$

so this does not correspond to a lattice point such as $\{3, \neg 47, 932\}$. The lattice point $\underline{p_0}$ *would lie below* the nodes $\{3\}$ $\{47\}'$ and $\{932\}$ in the lattice *if the lattice permitted it*, and not above them. Since $\{3\}$, $\omega - \{47\}, \{932\}$ are already singleton or co-singleton sets, and hence notional atoms or co-atoms (primes), the meet $\underline{p_0} \equiv \{3\} \wedge \{47\}' \wedge \{932\}$ does not lie in L_0; however, it will immediately be embedded in the next lattice up in the recursive sequence, $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ obtained by lowering the floor [see Sec. 1.4 above].

Notation

I denote a set of statements by $\overline{p_k}$. This corresponds

1. To a filter defined on the lattice and denoted p_k
2. To a lattice point, which is the minimum point of the filter, and denoted $\underline{p_k}$.

We aim to recursively generate a sequence of statements $\left\{ \overline{p_k} \right\}$ that act as successive approximations to a statement $\overline{q} = \lim_{k \to \infty} \overline{p_k}$ that defines a set $\underline{q}$ corresponding to a generic filter (set) *q*.

<u>Rule 1</u>

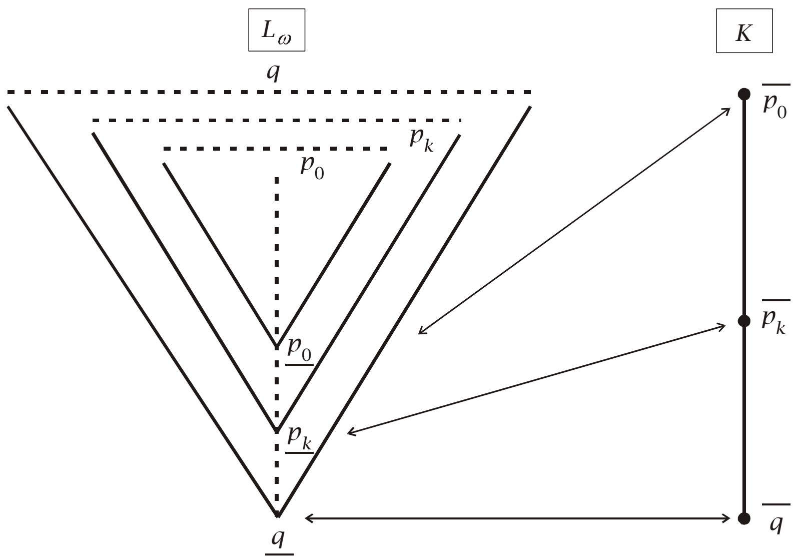
If $\overline{S_k} \in \overline{p_{k-1}}$ then $\overline{p_k} = \overline{p_{k-1}}$. Since $Q = \lim_{k \to \infty} \overline{p_k}$ this entails $\overline{p_k} \subseteq \overline{q}$.

Example

Let $\overline{p_0} = \left\{ \overline{3} \in \overline{q},\ \overline{47} \notin \overline{q},\ \overline{932} \in \overline{q} \right\}$. Suppose $\overline{S_1} \equiv \overline{3} \in \overline{q}$ then $\overline{p_1} = \overline{p_0}$.

[1] See, for example, Wolf [2005] p.236, where he uses curly brackets when defining a *condition*.

This rule serves to ensure that p_k lies in the filter generated by *q*. We have: $\underline{q} \le \underline{p}$ iff $\overline{p} \subseteq \overline{q}$. (Note, lattice points of the sequence $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ on the left of this, statements in the language *K* on the right, and the relation is inverted.)



On the left we have a supraset relation, $\underline{q} \subseteq \underline{p_k} \subseteq \underline{p_0}$ <u>in the sequence of lattice extensions</u> whereas on the right we have a subset relation $\overline{q} \supseteq \overline{p_k} \supseteq \overline{p_0}$ <u>in the language</u>.

<u>Rule 2</u>

If $\overline{S_k} = \neg\overline{S_j}$ where $\overline{S_j} \in \overline{p_{k-1}}$, then

1. $\overline{p_k} \vdash S_j$ (That is, $\overline{p_k} \vdash \neg\overline{S_k}$.)

2. $\overline{p_k} \Vdash \neg\overline{S_k}$ and $\overline{S_k} \notin \overline{p_k}$. This also means $\overline{S_k} \notin \overline{q}$.

This rule prevents $\overline{p_k}$ and $\overline{q}$ from being inconsistent sets of conditions. This in turn means that the lattice points to which they correspond cannot be names of the **0** of the lattice; that is, neither $\underline{p_k} \neq \mathbf{0}$ nor $\underline{q} \neq \mathbf{0}$. This means that $\underline{q}$, which is another name for the set $\{g\}$ is a point lying in the neighbourhood of **0** but distinct from **0** and not a member of any countably infinite lattice; it lies on the *boundary* between the neighbourhood of **0** and an ultrafilter defined by sequence of lattices: $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$. The correspondent generic filter (set) *q* lies on the boundary between the countably infinite and the uncountably infinite in the lattice $\mathbf{2}^\omega$, which is the Cantor set.

At any finite stage the set $\overline{p_k}$ is finite (we say “contains finite information”) so there are statements $\overline{S}$ of the form $\overline{m} \in \overline{q}$ or $\overline{m} \notin \overline{q}$ that are undecided at this stage. So we need a rule for deciding these statements as they come up in the recursive sequence of statements.

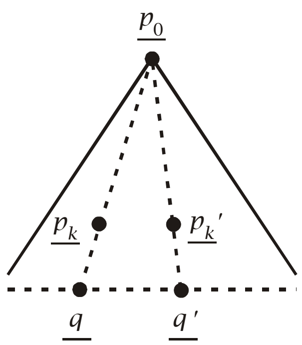
Rule 3

If $\neg\overline{S_k} \notin \overline{p_{k-1}}$ then at the stage $k - 1$ $\overline{S_k}$ has not yet been decided. (This means, $\overline{p_{k-1}} \not\Vdash \overline{S_k}$ and $\overline{p_{k-1}} \not\Vdash \neg\overline{S_k}$.) Then $\overline{p_k} = \overline{p_{k-1}} \cup \overline{S_k}$. This entails $\overline{p_k} \Vdash \overline{S_k}$ and $\overline{q} \Vdash \overline{S_k}$

This simply says that if at stage $k - 1$ we have $\overline{S_k}$ undecided, then we decide $\overline{S_k}$ at the stage k by adding it to the filter p_k.

1.8 Theorem, generic sets cannot be constructed by finite information

Together these three rules applied to a point $\underline{p_0}$ create an *ideal* at $\underline{p_0}$. The generic set $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$ lies at on a non-compact path of actually ω lattice points distant from $\underline{p_0}$. There are many such paths winding through the ideal at $\underline{p_0}$, and the recursive sequence $\overline{S}_1, \overline{S}_2, \ldots, \overline{S}_k, \ldots$ picks out one of these and homes in upon the limit point $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$. There are more than ω such paths; the number of paths is an ordinal β such that $\aleph_0 < |\beta| \le 2^{\aleph_0}$.



It can be seen immediately that the supposition that the generic set $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$ can be constructed at any finite stage leads to a contradiction. Suppose $\overline{q} = \overline{p_k}$. Then since $\overline{p_k}$ is finite there must be some $n \in \mathbb{N}$ such that both $n \notin q$ and for the statement $\overline{S} \equiv (\overline{n} \notin \overline{q})$ we have $(\overline{n} \notin \overline{q}) \notin \overline{p_k}$. That is $\neg(\overline{n} \in \overline{q}) \notin \overline{p_k}$ and $\overline{S} \equiv (\overline{n} \notin \overline{q})$ is undecided at $\overline{p_k}$. Then by Rule 3 above we have $\overline{q} \Vdash (\overline{n} \in \overline{q})$, which is $(\overline{n} \in \overline{q}) \in \overline{q}$; hence $n \in q$. So we obtain $n \in q$ and $n \notin q$, a contradiction. Hence, the supposition $\overline{q} = \overline{p_k}$, that q is constructed at some finite stage is false. This is because q is defined by an <u>inductive procedure</u> that take us out of the sequences of lattices $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ of countably infinite lattice points – that is, it enables us to *transcend* it – and *no analytic process could do that.* So no generic set is “constructible” *in the sense of recursively enumerable*, though it is “definable” in a wider sense. [See also Chapter 11 for further discussion of the differences between “definable”, “constructible” and “recursive”] The assumption that a generic set is recursively enumerable leads to a contradiction. Now we see that the generic sequence $\overline{q} = \lim_{k\to\infty} \overline{p_k}$ represents an inductive rule for continuing a series of lattice points indefinitely below any given lattice point $\underline{p_0}$, and the contradiction arises from assuming that $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$ can be completed at some definite point, which is the same as assuming that a generic set $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$ is completely defined by a lattice point $\underline{p_k}$. In essence the “non-constructibility” of a generic set arises from the fact that no countably infinite lattice is atomic, which entails that there are always lattice points lying below any putative atom, and the contradiction arises from assuming that a finite set of conditions $\overline{p_k}$ defines an atom. Only an atom could define $\{g\} = \underline{q} = \lim_{n\to\infty} \underline{p_n}$, and no such atom can be reached by any finite, that is effectively computable (recursive) procedure.

As already indicated the generic set q is defined by a complete sequence of lattice points $\underline{q} = \lim_{k\to\infty} \underline{p_k}$ that must be infinite in length (therefore, not compact) and also contain a given set of partial information represented by the finite set $\overline{p_0}$. The contradiction immediately arises from assuming that a given finite set $\overline{p_k}$ is sufficient to define $\overline{q}$, which is equivalent to making $\omega = n$ where n is finite. In $\overline{p_k}$ not all the information is given so there must be statements of the form $\overline{n} \in \overline{q}$ that are true but not decided, but in $\overline{q}$ all such statements are decided. So we take a generic set q for which we definitely have $n \in q$ but this is not decided by $\overline{p_k}$ and by the rules of the construction of q, this entails that $n \notin q$. So it is immediately obvious that <u>if there exists a generic set</u> in which every statement is decided, then this generic set cannot be defined by a finite amount of information.

2 Transcendental numbers

2.1 Definition, algebraic

A real or complex number is said to be algebraic if it is the zero of a polynomial with integer coefficients.

2.2 Result

Every algebraic number α is the zero of some irreducible polynomial f that is unique up to constant multiple. The degree of α, denoted $\partial\alpha$ is the degree of the polynomial f

2.3 Liouville's theorem

Let α be an algebraic number with degree $n > 1$. Then there exists a $c = c(\alpha)$ such that $\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^n}$ for all rationals $\frac{r}{s}, s \in \mathbb{N} > 0$.

Proof

Let $f(x)$ be an irreducible polynomial with root α. The mean value theorem is $f'(\xi) = \frac{f(b) - f(a)}{b - a}$, $a \le \xi \le b$. On substituting $\frac{r}{s} = b$ we obtain

$$f'(\xi) = \frac{f(\alpha) - f\left(\frac{r}{s}\right)}{\alpha - \frac{r}{s}}, \ \alpha \le \xi \le \frac{r}{s}$$

Hence, since $f(\alpha) = 0$,

$$-f\left(\frac{r}{s}\right) = f'(\xi)\left(\alpha - \frac{r}{s}\right)$$

If $\left|\alpha - \frac{r}{s}\right| \ge 1$ the result is trivially true because $\frac{1}{s^n} \le 1 \le \left|\alpha - \frac{r}{s}\right|$. Then, for the non-trivial case, suppose $\left|\alpha - \frac{p}{q}\right| < 1$. Then $\xi \le \frac{p}{q} - \alpha$, $|\xi| \le \left|\alpha - \frac{p}{q}\right| - 1$ and $|\xi| < 1 + |\alpha|$. As $|\xi| < 1 + |\alpha|$ we have ξ is close to α. At α we have $f'(\alpha) = 0$, so $f'(\xi) \to 0$ as $\xi \to \alpha$. That means that $|f'(\xi)| \to 0$, which means that for given ξ there is a $c = c(\alpha) > 0$ such that $|f'(\xi)| < \frac{1}{c}$. Thus $|f'(\xi)| < \frac{1}{c}$ where $c = c(\alpha) > 0$. From $-f\left(\frac{r}{s}\right) = f'(\xi)\left(\alpha - \frac{r}{s}\right)$ we obtain

$$\left|f\left(\frac{r}{s}\right)\right| = |f'(\xi)|\left|\left(\alpha - \frac{r}{s}\right)\right|$$

Then substituting $|f'(\xi)| < \frac{1}{c}$ gives

$$\left|f\left(\frac{r}{s}\right)\right| = \frac{1}{c}\left(\left|\alpha - \frac{r}{s}\right|\right)$$

This gives

$$\left|\alpha - \frac{r}{s}\right| < c\left|f\left(\frac{r}{s}\right)\right|$$

But *f* is irreducible, hence $f\left(\frac{r}{s}\right) \neq 0$ and the integer $\left|q^n f\left(\frac{r}{s}\right)\right| \geq 1$. Hence, $\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^n}$ as required.[2]

2.4 (+) Claim

The condition $\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^n}$ is a forcing condition on a generic sequence; any individual g constructed as a result of a generic sequence subject to this condition defines an atom $\{g\}$ and a generic set *q*.

2.5 Existence of transcendental numbers

Let $\xi = \sum_{n=1}^{\infty} 10^{-n!} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^{3!}} + \frac{1}{10^{4!}} + \ldots = 0.110001000000000000000001000\ldots$.

Let

$r_j = 10^{j!}\sum_{n=1}^{j} 10^{-n!}$ $\qquad s_j = 10^{j!}$ $\qquad (j = 1,2,3,\ldots)$

Then r_j, s_j are relatively prime rational integers.

$$\left|\xi - \frac{r_j}{s_j}\right| = \sum_{n=j+1}^{\infty} 10^{-n!} < 10^{-(j+1)!}\left(1 + 10^{-1} + 10^{-2} + \ldots\right) = \frac{10}{9}\cdot\frac{1}{10^{(j+1)!}} = \frac{10}{9}\cdot\frac{1}{10^{(j+1)j!}} = \frac{1}{9}\cdot\frac{1}{10^{j\cdot j!}} < \frac{1}{\left(s_j\right)^j}$$

By Liouville's theorem, if ξ is algebraic, then $\xi > \frac{1}{\left(s_j\right)^j}$, so ξ must be transcendental.

2.6 (+) Theorem

$\{\xi\}$ is an atom in the Cantor set $\mathbf{2}^{\omega}$, and $q = \text{filter}\{\xi\}$ is a generic set.

<u>Proof</u>

Let $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_k, \ldots$ be a recursive enumeration of statements such that $\overline{S_n} \equiv \overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \bar{q}$.

[2] Barker [1975] states that an explicit value for *c* is given by $\frac{1}{c} = n^2\left(1 + |\alpha|\right)^{n-1} h$ where *h* denotes the height of α, which is the maximum of the absolute values of the coefficients of *f*.[2]

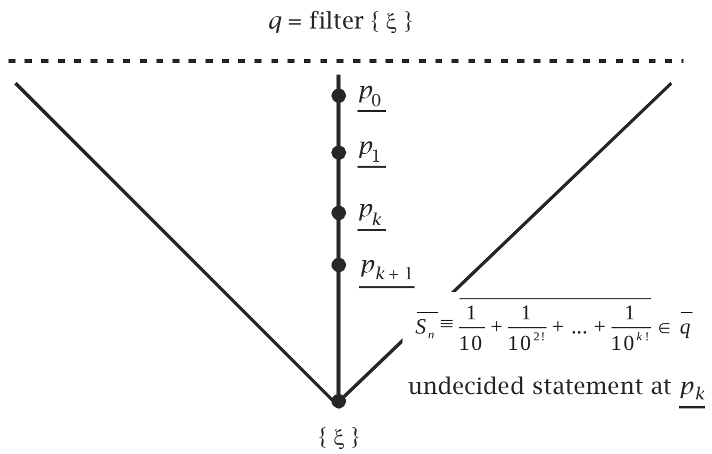
Let

$p_0 = \{0\}$ $\qquad \overline{p_0} \equiv \left\{\overline{\{0\} \in q}\right\} \equiv \left\{\overline{0} \in \overline{q}\right\}$

$p_1 = \left\{\frac{1}{10}\right\}$ $\qquad \overline{p_1} \equiv \overline{p_0} \cup \left\{\overline{\frac{1}{10}} \in \overline{q}\right\}$

$p_{k+1} = \left\{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}\right\}$ $\qquad \overline{p_{k+1}} \equiv \overline{p_k} \cup \left\{\overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \overline{q}\right\}$

That is $\overline{p_k} \equiv \overline{p_{k-1}} \cup \overline{S_k}$. The following diagram illustrates this proof.



Let an ideal be defined by the inductive rule: -

$$\overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{k!}}} \in \overline{q} \Rightarrow \overline{\frac{1}{10} + \frac{1}{10^{2!}} + \ldots + \frac{1}{10^{(k+1)!}}} \in \overline{q}$$

Then $\overline{q} = \overline{p_k}$ where $k \in \mathbb{N}$ entails that ξ is algebraic. Then by Liouville's theorem $\xi > \frac{1}{(s_k)^k}$ where $s_k = 10^{k!}$. But by the preceding result $\xi < \frac{1}{(s_k)^k}$. Hence $\overline{q} \neq \overline{p_k}$ and q is a generic set.

Remark

ξ could be a binary number as well. That is, it could be interpreted as a binary expansion where $10_2 = 2_{10}$. That displays it explicitly as a member of the Cantor set via the notion of a binary tree.

Then $\{\xi\}$ is an element of the boundary of $\mathbf{2}^{\omega}$. [For further discussion of this point, see Sec. 3 below]

Burger and Tubbs [2004] (p 9 – 11) make the following comments about proofs concerning the transcendence of numbers. (1) The "Fundamental principle of number theory" is that there are no integers between 0 and 1. (2) If a number α that is transcendental is assumed to be algebraic then one can construct an integer N that violates the fundamental principle of number theory. (3) With specific regard to Liouville's theorem: if we find an α that has an infinite number of amazingly good rational approximations, approximations so good that they violate $\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^n}$ for any choice of *n* and *c*, then α must be transcendental.

2.7 Observation

In the proof that ξ is transcendental we see that at each stage *k* we construct a rational approximation to ξ such that $\xi < \frac{1}{(s_k)^k}$, which means that ξ lies between the floor of the lattice and 0. Hence, ξ is a number that lies in the interval $[0,1]$ lies in the neighbourhood of 0 whenever that interval is subject to a partition into ω parts. The sequence $\underline{p_0} \to \underline{p_1} \to \ldots \to \underline{p_k} \to \ldots$ is an attempt to "home in" upon ξ and is equivalent to the method of descent, starting in the neighbourhood of 1 and moving by a recursive process in the direction of ξ. However, ξ can never be reached by such a process, and lies *on the boundary* between the neighbourhood of 0 and the next notional atom away from 0 in the partition of $[0,1]$ into ω parts.

2.8 The "Fundamental principle of number theory"

As remarked above, Burger and Tubbs [2004] state that the "Fundamental principle of number theory" that there are no integers between 0 and 1. In Kantian terminology this qualifies as a *synthetic a priori* principle [Defined, Chap. 2 / 2.1]: it is impossible to see how such a law could be derived from an analytic partition of space; on the contrary, any analytic partition of space must rest upon this as a principle.

2.9 Definition, Liouville number

A *Liouville number* is any real number ξ that possesses a sequence of distinct rational approximations $\frac{p_n}{q_n}\,(n = 1,2,3,\ldots)$ such that $\left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^{\omega_n}}$ where $\lim(\sup \omega_n) = \infty$. It is proven above [2.5] that all Liouville numbers are transcendental.

Barker [1975] remarks on the application of Liouville's theorem to transcendental numbers that "... any non-terminating decimal in which there occur sufficiently long blocks of zeros, or any continued fraction in which the partial quotients increase sufficiently rapidly, provides an example." (p.2) He comments that in the Cantorian sense "almost all" numbers are "transcendental"; they cannot be constructed recursively.

Hermite proved in 1873 that the number e is transcendental. The presentation of his theorem that follows here is due to Baker [1975].[3]

2.10 Hermite's theorem

e is transcendental.

Lemma 1

Let $f(x)$ be any polynomial of real coefficients of degree m. Let

$$I(t)=\int_0^t e^{t-u}f(u)\,du$$

where t is an arbitrary complex number and the integral is taken over the line joining 0 to t. Integration by parts gives

$$\begin{aligned}I(t)&=\int_0^t e^{t-u}f(u)\,du\\&=\left[f(u)\cdot -e^{t-u}\right]_0^t+\int_0^t f^{(1)}e^{t-u}\\&=-f(t)+e^t\cdot f(0)+\int_0^t f^{(1)}e^{t-u}\end{aligned}$$

and repeated integration by parts gives

$$I(t)=e^t\sum_{j=0}^m f^{(j)}(0)-\sum_{j=0}^m f^{(j)}(t)$$

Lemma 2

Let $\overline{f}(x)$ denote the polynomial that is obtained from f by replacing each coefficient in f with its absolute value. Then

$$|I(t)|\le\int_0^t\left|e^{t-u}f(u)\right|du\le|t|e^{|t|}\overline{f}(|t|)$$

Proof of the theorem

1. Suppose e is algebraic. Then there exist integers $q_0,q_1,\dots,q_n,\ n>0$ such that

$$q_0+q_1e+\dots+q_ne^n=0.$$

Let

$$J=q_0I(0)+q_1I(1)+\dots+q_nI(n)$$

where $I(t)$ is defined as in the lemma and $f(z)=z^{p-1}(z-1)^p\dots(z-n)^p$ and p is a large prime. Substituting $I(t)=e^t\sum_{j=0}^m f^{(j)}(0)-\sum_{j=0}^m f^{(j)}(t)$ we obtain: -

[3] Baker's treatment could be said to be "light". A fuller treatment is in Burger and Tubbs [2004], though that is not more perspicuous. For the purposes here it is not required to clarify every inference, since the aim here is soley is to show that Hermite's proof constructs e as a generic set.

$$I(t) = q_0\left\{\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t)\right\} + q_1\left\{e\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(1)\right\} + \ldots + q_n\left\{e^n\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(n)\right\}$$
$$= q_0\left\{-\sum_{j=0}^{m} f^{(j)}(t)\right\} + q_1\left\{-\sum_{j=0}^{m} f^{(j)}(1)\right\} + \ldots + q_n\left\{-\sum_{j=0}^{m} f^{(j)}(n)\right\}$$
$$= -\sum_{j=0}^{m}\sum_{j=0}^{n} q_k f^{(j)}(k)$$

where $m = (n+1)p - 1$. We have $f^{(j)}(k) = 0$ if $j < p$ and $k > 0$ or if $j < p-1$ and $k = 0$. Hence, for all *j, k* except $j = p-1$ and $k = 0$ we may say that $f^{(j)}(k)$ is an integer divisible by $p!$. Furthermore,

$f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p$

from which it follows that if $p > n$, we have $f^{(p-1)}(0)$ is an integer divisible by $(p-1)!$ but not by $p!$ Then, if $p > |q_0|$, we have *J* is a non-zero integer divisible by $(p-1)!$. hence $|J| \geq (p-1)!$

This establishes one estimate for $|J|$ which is based on the first lemma and applies on the assumption that *e* is algebraic.

2. To obtain another estimate, note that $\overline{f}(k) \leq (2n)^m$ and this combined with

$\left|I(t)\right| \leq \int_0^t \left|e^{t-u} f(u)\right| du \leq |t| e^{|t|} \overline{f}(|t|)$

from the second lemma, gives

$|J| \leq |q_1| + \ldots + |q_n| n e^m \overline{f}(n) \leq c^p$

where *c* is independent of *p*.

3. If *p* is sufficiently large the two estimates are contradictory, whence *e* cannot be a root of an irreducible algebraic polynomial.

2.11 (+) Theorem

$\{e\}$ is an element in the Cantor set $\mathbf{2}^{\omega}$, and $q = \text{filter}\{e\}$ is a generic set.

Proof

Let $q = \text{filter}\{e\}$ be the consequence of everything that follows from the assumption that *e* is the zero of a function *f*; that is $f(e) = 0$. That is $\overline{\phi} \in q$ iff $\overline{f(e)} = \overline{0} \models \overline{\phi}$.

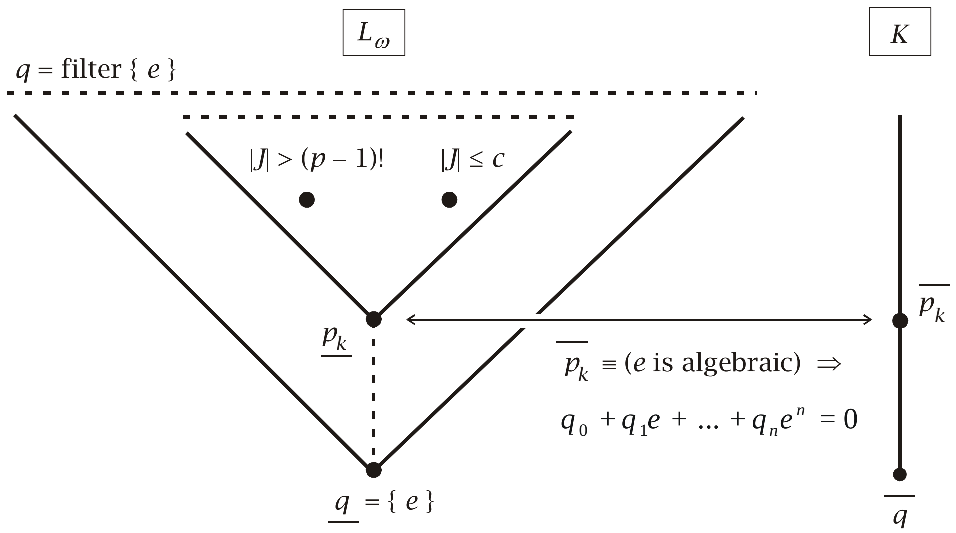
In the proof of Hermite's theorem we assume that *e* is algebraic. This is equivalent to the assumption that *f* is an algebraic function; hence there exist integers $q_0, q_1, \ldots, q_n, n > 0$ such that $q_0 + q_1 e + \ldots + q_n e^n = 0$. This information is encoded in a finite set $\overline{p_k}$, and the assumption is that $\overline{q} = \overline{p_k}$. This assumption is shown in Hermite's proof to *decide* two contradictory statements. Given

$J = q_0 I(0) + q_1 I(1) + \dots + q_n I(n)$

where $I(t) = \int_0^t e^{t-u} f(u)\,du$, we have

1. $|J| \geq (p-1)!$
2. $|J| \leq c^p$.

Since this is a contradiction violating the Fundamental Principle of Number Theory, these statements cannot be so decided. Hence $\overline{q} \neq \overline{p_k}$ and q is a generic set.



To show that $\{e\}$ is in the Cantor set, we invoke the principle that all real numbers may be represented by infinite sequences of 1s and 0s.

The assumption that e is algebraic is equivalent to the claim that $\{e\}$ can be constructed at a finite ordinal level $k \in \mathbb{N}$; but whatever finite ordinal level is chosen, corresponding to a lattice point $\underline{p_k}$, there is a contradictory statement at that level.

2.12 Generic sets and the solution to the halting problem

Let q be the filter (generic set) defined by $\rho(n)$ = the productivity of the most productive n-state Turing machine. This corresponds to a complete set of forcing conditions $\overline{q}$ since every statement of the form S_n: $\rho(\overline{n}) = \overline{m}$ is decided in $\overline{q}$. Now suppose this information is encoded in a finite set $\overline{p_k}$, so that $\overline{q} = \overline{p_k}$ for some $k \in \mathbb{N}$. Then the following statements are decided at this level. [Chap. 3 / 1.5]

S_0	$\rho(1) = 1$	S_2	$\rho(n+1) > \rho(n)$
S_1	$\rho(47) \geq 100$	S_3	$\rho(n+11) \geq 2n$

These all correspond to lattice points in the lattice $\mathbf{2}^{<\omega}$ and are contained in the filter p_k. They constitute a set of forcing conditions $\overline{p_k} \Vdash S_0 \wedge S_1 \wedge S_2 \wedge S_3$ and lead to a contradiction. Hence $\overline{q} \neq \overline{p_k}$ and q is a generic set. By mathematical induction we know that the filter q can be defined. Hence, mathematical induction is a synthetic principle of reasoning and Poicaré's thesis is upheld.

3 "Location" of the generic set

Consider the real number e that is defined by a generic sequence in a language K describing a sequence of generic filter extensions $L_0, L_1, \ldots, L_k, L_{k+1}, \ldots$ all of which are algebraic and hence isomorphic to $F(\omega) \cong \mathbf{2}^{<\omega}$, the filter of a finite subsets of ω. Then e is a transcendental number that is the individual belonging to an atom $\{e\}$ of some larger lattice L_ω that is also defined by the same generic sequence that defines e. We see that $\{e\}$ cannot be an atom of the Cantor set $\mathbf{2}^{\mathbb{N}_\infty} \cong \mathbf{2}^{\omega} \cong \{0,1\}^{\omega}$ even though it is a member of this set. This follows from the 1-point compactification: $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda$. The atoms of $\mathbf{2}^{\mathbb{N}_\infty}$ are either the singleton sets of natural numbers $\{n\}, n \in \mathbb{N}$ or $\lambda = \{\mathbb{N}\}$, and we have $e \neq n, n \in \mathbb{N}$ and $e \neq \lambda$. So the generic sequence for e defines a (transcendental) real number that is not an <u>atom</u> of the Cantor set $\mathbf{2}^{\mathbb{N}_\infty} \cong \mathbf{2}^{\omega}$ though it is in correspondence to an <u>element</u> of it. It is an atom of lattice generated by the Cantor set – every member of the Cantor set is an atom of this lattice, which is the lattice generated by the continuum of real numbers. The atoms are singletons of real numbers.

In logical terms the generic sequence has defined an element of a non-standard model of arithmetic. (See Boolos and Jeffrey [1980] Chap. 17) Non-standard models append to the sequence of natural numbers elements that are not natural numbers; for example, $\mathbb{N} = \{0,1,2, \ldots, \ldots, a,b,c\}$ where *a, b, c* are not natural numbers. Non-standard models violate ω-consistency [Defined Chap.9 / 1.10]. In fact, we see that $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda = \{0,1,2,\ldots,\lambda\}$ is itself a *non-standard model of arithmetic*. This is the fundamental reason why first-order logic is not categorical and obeys the upwards and downwards Löwenheim-Skolem theorems, because the language permits the definition of non-standard models, and indeed essentially so, since we use it to define transcendental numbers.

The only member of $\mathbb{N}_\infty = \{0,1,2,\ldots,\lambda\}$ that could be defined as an individual by any generic sequence is $\lambda = \{\mathbb{N}\}$ itself. We will see in Chapter 9 that Gödel's theorem itself defines $\lambda = \{\mathbb{N}\}$ and hence demonstrates that $\lambda = \{\mathbb{N}\}$ cannot be recursively enumerated *as an actual infinity*. Its members can be recursively enumerated, but as an actually completed totality it cannot be constructed – which is self-evident, because *we could never have done with counting all the natural numbers* and a machine couldn't do this either. Unlike machines we can *conceive* of this totality even if we cannot count all its members.

CHAPTER 9

Incompleteness and Poincaré's thesis

1 Gödel's first incompleteness theorem

1.1 Gödel's first incompleteness theorem

There is a formula Q of a sufficiently strong [defined1.6 below] theory K that is not provable.

It is the task of this section to explain the terms involved in this theorem, and to demonstrate its proof. Gödel's theorem is the main counter-example to formalism and establishes categorically the validity of Poincaré's thesis. In order to demonstrate this, we require a detailed and annotated version of the proof of this theorem.

Notation

In what follows we use the Greek letters, $\xi, \zeta, \chi, \ldots$ to denote variables. We do this in order to maintain a clear distinction between formulae and variables. In the discussion that follows we shall have a need to introduce several kinds of formula, so we use the following as variables for predicates: $\phi, \psi, \ldots, X, Y, \ldots$.

1.2 Substitution of terms

Consider a formula $\phi(\xi)$ where ξ is a variable. Then we may *substitute* a constant a for ξ to obtain $\phi(a)$.

Definitions, term, μ – operator

A *term* is any formula that may be substituted for a variable. Terms include: (1) Constants, *a, b*, ...; (2) Other variables: $\zeta, \chi, \ldots$

(3) Formulas involving other variables: -

3.1 Unbound variables: $\psi(\zeta)$.

3.2 Definite descriptions: -

3.21 "The (unique) *y* such that ψ": $(\iota\zeta)\psi(\zeta)$.

3.22 "The least *x* such that ψ": $(\mu\zeta)\psi(\zeta)$.

This involves a μ – operator .

1.3 Predicates and sentences

Formulae are divided into predicates, terms and sentences. A sentence (or proposition) arises from the combination of a predicate with a term. Suppose ϕ is a predicate. Then to form a sentence we must adjoin to it a term *a* to obtain an expression of the form $A = \phi(a)$. The predicate may be viewed as an uncompleted sentence – a sentence with a gap in it that can be filled by a term. Thus we may write $\phi = \phi(\xi)$ where the unbound variable ξ represents the gap in the predicate. A sentence is a completed predicate. To make a sentence one must substitute for an unbound variable in the predicate either a term or a bound variable. Thus $\phi(a)$ and $(\forall\xi)\phi(\xi)$ are sentences. In first-order predicate calculus all sentences arise from the completion of predicates and there are no exceptions. However, we sometimes wish to introduce a label for sentences that does not express their internal structure. In that case we may use $A, B, C, \ldots$. We shall use *P* and *Q* to denote very specific sentences that shall be defined below.

1.4 Numbers and numerals

The theory under consideration is a sufficiently strong [1.6 below], first-order theory, *K*. Such a theory is capable of expressing number-theoretical statements. For example, let: $\phi = \phi(\zeta) \equiv \xi \text{ is even}$. In this expression $\phi(a)$, *a* denotes an individual that is even, such as 2. However, *a* is <u>not</u> the number 2 but denotes the number 2. A term that denotes a number is called a *numeral*. In most contexts it is not necessary or normal to distinguish numerals from numbers. Here, it is important to do so. If $n \in \mathbb{N}$ is a natural number then $\bar{n}$ is the numeral that denotes *n*. For example, $\bar{2}$ denotes 2. If $\phi(\zeta)$ is the predicate denoting the property of being even, then the expression "2 is even" is represented by $\phi(\bar{2})$. Thus, $\xi, \zeta, \chi, \ldots$ are used here to range over <u>numerals</u>, as opposed to numbers. We also use $\bar{n}, \bar{m}, \ldots$ as variables in this sense.

1.5 Gödel numbering

Gödel numbering is a mapping from formulae to natural numbers.

$$\text{Gn}\begin{cases}\text{Formulae} \to \mathbb{N} \\ \phi \to \text{Gn}(\phi)\end{cases}$$

The notation $\text{Gn}(\phi)$ is useful because it emphasises the role of Gödel numbering as a function transforming a formula into a natural number; however, we shall find it convenient to introduce another notation for this: $\ulcorner\phi\urcorner \equiv \text{Gn}(\phi)$. Thus $\ulcorner\phi\urcorner$ represents the Gödel number of an arbitrary formula ϕ. It is possible to define several Gödel functions by different specifications of the way in which numbers are assigned to particular symbols of the language, but every such definition must contain two steps.

(1) Each logical constant, variable, predicate letter, functional symbol or constant is assigned a distinct natural number. No two such symbols are mapped to the same number.

(2) Given the expression $u_1, u_2, ..., u_r$, we define its Gödel number to be $\ulcorner u_1, u_2, ..., u_r \urcorner = 2^{\ulcorner u_1 \urcorner} \times 3^{\ulcorner u_2 \urcorner} \times ..... \times p_r^{\ulcorner u_r \urcorner}$ where p_r is the *i*th prime number.

The function, Gn, is a one-one mapping, meaning, that given an formula *A* we may assign to it one and only one natural number *n*, and given any natural number *n* we may apply the inverse of Gn to determine what, if any, expression or symbol it represents. The Gödel function allows us to *arithmetize* logic.

1.6 Definition, Sufficiently strong

A logic *K*, is *sufficiently strong*, if the two concepts symbolised by: -

$\mathrm{Pf}(x,y)$ $\mathrm{Sub}(x,y,z)$

are recursive (or primitive recursive) and can be represented by a Gödel number. $\mathrm{Pf}(x,y)$ is interpreted to mean "*x* is the Gödel number of a proof of the formula with Gödel number *y*", and $\mathrm{Sub}(x,y,z)$ is the Gödel number of the formula obtained from the formula with Gödel number *x*, by substituting for the variable with Gödel number *y*, if it occurs free in *x*, the formula with Gödel number *z*. In the expression $\mathrm{Pf}(x,y)$, *x* and *y* are variables ranging over numbers, and not formulae or numerals.

1.7 Concerning the proof that *K* is sufficiently strong

To prove that *K* is sufficiently strong we have to prove that $\mathrm{Pf}(x,y)$ and $\mathrm{Sub}(x,y,z)$ are recursive. In a rigorous proof we begin with the notions of addition and multiplication, which are in fact primitive recursive, and by successive stages we construct formulae which are recursive and which ultimately define the notions $\mathrm{Pf}(x,y)$ and $\mathrm{Sub}(x,y,z)$. The formulae by which we define these successive concepts are formulae of quantification theory with identity; we therefore require as a preliminary for Gödel's theorem, a proof of a theorem that shows that the use of the symbols of quantification theory, and also bounded quantifiers and μ – operators [1.2 above], does not take us out of the class of recursive or primitive recursive functions. Such a theorem does in effect give us the "glue" with which to stick the formulae of our definitions together. The definition of each of the concepts involved proceeds in two steps: (i) we provide a formula which, using only notions already defined, expresses the concept in question; (ii) we prove, or verify, that the formula thus obtained is recursive. In many cases this second part follows directly; in others, we have to apply either recursion of "course-of-values recursion". In course-of-values recursion the definition of a function $f(x_1,....,x_n,y+1)$ depends on several of all values of $f(x_1,....,x_n,u)$ where $u \le y$. Here we will not provide precise proofs that $\mathrm{Pf}(x,y)$ and $\mathrm{Sub}(x,y,z)$ are recursive. For $\mathrm{Pf}(x,y)$ and $\mathrm{Sub}(x,y,z)$ to be definable by recursive functions a theory, *K*, must possess the following properties: -

1. The theory is able to *represent* [see Chapter 11] the functions $+$ and $\times$.
2. The underlying logic is first-order quantification theory with identity.
3. The Gödel function is defined for the theory.

These three properties together confer the property of sufficient strength on *K*.[1] *K* is also such that: -

4. The axioms of *K* are recursive, and hence *K* is axiomatizable.

It is assumed that *K* possesses these four properties, so *K* is sufficiently strong and recursively axiomatisable. Another statement of Gödel's Theorem is, then, the following: -

(I) If a theory, *K*, is consistent, axiomatizable and sufficiently strong, then it is incomplete.

(II) The theory *K* is axiomatizable and sufficiently strong; we also assume that it is consistent.

1.8 Getting started on the proof

Let $\ulcorner X \urcorner$ denote the Gödel number of $X = X(\xi)$.

Let $\ulcorner Y \urcorner$ denote the Gödel number of $Y = Y(\zeta)$.

Let $\ulcorner \Gamma \urcorner$ denote the Gödel number of the concatenation of a sequence of formulae: $\Gamma = \phi_1 \circ \phi_2 \circ ... \circ \phi_n$.

Let $\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)$ denote the Gödel number of a proof of *X* from the sequence Γ. Here proof is a relation between a sequence Γ and a formula *X*. It may be written $\Gamma \vdash X$. In the usual notation $\Sigma \vdash X$ we have Σ is a set of premises. In $\Gamma \vdash X$, Γ represents the whole finite path, from some minimum lattice point, Σ, below which it is not possible to proceed, to *X*. We have, $\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner) = \ulcorner \Gamma \vdash X \urcorner$. The proof relation $\vdash$ is assumed to be that of the language *K* in which

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[1] For a more formal presentation of what *sufficient strength* means, we have the following result from Mendelson [1970]: Result, general applicability of the Gödel-Rosser theorem: Let *K* be the first-order theory of arithmetic for which the Gödel-Rosser theorem holds. Let K' be another theory with the same symbols as *K*. Suppose

1. Every recursive relation is expressible in K'.
2. The set of Gödel numbers of proper axioms of K' is recursive
3. For any wff $A(x)$ and any natural number *k*: $\vdash_{K'} A(\bar{0}) \wedge A(\bar{1}) \wedge ... \wedge A(\bar{k}) \supset (\forall x)(x \le \bar{k} \supset A(x))$ (This is complete induction.)
4. For any natural number *k*, $\vdash_{K'} x \le \bar{k} \wedge \bar{k} \le x$

Then the Gödel-Rosser theorem [2.4 above] holds for K'.

the proof is written. When we wish to emphasise the presence of *K* we write this $\Gamma \vdash_K X$. Here the expressions "language" and "first-order theory" may be used interchangeably. Let $\mathrm{Sub}(\ulcorner X \urcorner, \ulcorner \xi \urcorner, \ulcorner Y \urcorner)$ be the Gödel number formula obtained from the formula $X = X(\xi)$ by substituting for the variable ξ (if it is free in *X*) the formula $Y = Y(\zeta)$. That is, $\mathrm{Sub}(\ulcorner X \urcorner, \ulcorner \xi \urcorner, \ulcorner Y \urcorner) = \ulcorner X(Y(\zeta)) \urcorner$[2].

1.9 The distinction between object- and meta-language

We are discussing the properties of an object language *K* within a meta-language, represented by natural language and equipped with the logic and reasoning of elementary number theory. We do not give this meta-language a name as such. The object language *K* is *sufficiently strong*. This entails that any recursive property or expression of the meta-language has a corresponding wff in the object-language. In the meta-language we have properties and individuals. The only individuals that concern us are *numbers*. The argument involves going backwards and forwards between the meta- and object-languages. Therefore, we have to be careful to distinguish those expressions that belong to the meta-language from those that belong to the object-language. An expression in either the object- or meta-language cannot be of mixed type. The general distinction is as follows: -

meta-language	object-language
numbers	numerals
	terms
number-theoretic properties	predicates
natural language statements	sentences
natural language logic	first-order logic

We need some devices to keep the distinction clear. We already have the distinction between Gödel numbers and the formulae they represent. Thus $\ulcorner X \urcorner$ is the number (meta-language) that represents the wff formula *X* (object-language) and conversely. Also $\bar{n}$ is the numeral (object-language) representing the natural number *n* (meta-language). We need an additional device to map back natural language statements in the meta-language to the wff in the object-language that represent them. For this we use the underscore. For example $\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)$ in the meta-language is read "$\ulcorner \Gamma \urcorner$ is the Gödel number of the sequence of formulae Γ that prove the formula *X*, whose Gödel number is $\ulcorner X \urcorner$". This natural language statement corresponds to the object-language formula $\Gamma \vdash X$. Therefore, we may write $\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \equiv \Gamma \vdash X$.

[2] This is very similar to the process familiar from mathematics of a change of variables.

1.10 Definition, ω - consistency

Let K be a first-order theory. Let $\bar{n}$ denote the numeral representing the number $n \in \mathbb{N}$. Suppose for every $\bar{n} \vdash_K \neg A(\bar{n})$ then $\nvdash_K (\exists x) A(x)$. Then K is said to be ω-consistent.

The effect of ω-consistency is to rule out non-standard models of arithmetic.[3]

1.11 Outline of proof

We define a particular sentence: -

$$P(\xi) \equiv (\forall \bar{x})\left(\neg\, \underline{\mathrm{Pf}\left(x, \mathrm{Sub}\left(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner\right)\right)}\right)$$

We let $k = \ulcorner P \urcorner$ - that is k is the Gödel number of P. Let

$$Q \equiv (\forall \bar{x})\left(\neg\, \underline{\mathrm{Pf}\left(\bar{x}, \mathrm{Sub}(k,k,k)\right)}\right)$$

and let $m = \ulcorner Q \urcorner$. We claim, provided the theory K is ω-consistent, that m is the Gödel number of a formula that is not a proof of itself.

1.12 Demonstration

To show this we merely have to work through the series of correspondences these definitions produce between formulae ϕ and their Gödel numbers $\ulcorner \phi \urcorner$ on the other.

Godel number, $\ulcorner \phi \urcorner$ / meta-language	Formula, ϕ / object-language
n, k,...	$\bar{n}$, $\bar{k}$, ... (numerals denoting numbers)
x, y, ... (variables for numbers)	$\bar{x}$, $\bar{y}$,... (variables for corresponding numerals)
$\ulcorner X \urcorner$	$X = X(\xi)$
$\ulcorner \Gamma \urcorner$	$\Gamma = \phi_1 \circ \phi_2 \circ \ldots \circ \phi_n$
$\mathrm{Pf}\left(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner\right)$	$\Gamma \vdash X$
$\mathrm{Sub}\left(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner\right)$	$X\left(X(\xi)\right)$
$\mathrm{Pf}\left(\ulcorner \Gamma \urcorner, \mathrm{Sub}\left(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner\right)\right)$	$\Gamma \vdash X\left(X(\xi)\right)$
	$\neg\, \underline{\mathrm{Pf}\left(\ulcorner \Gamma \urcorner, \mathrm{Sub}\left(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner\right)\right)} \equiv \Gamma \nvdash X\left(X(\xi)\right)$
$k = \ulcorner P \urcorner$	$P(\xi) \equiv (\forall \bar{x})\left(\neg\, \underline{\mathrm{Pf}\left(x, \mathrm{Sub}\left(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner\right)\right)}\right) \equiv \nvdash X\left(X(\xi)\right)$
$\mathrm{Sub}(k,k,k)$	$P\left(P(\xi)\right)$
$\mathrm{Pf}\left(\ulcorner \Gamma \urcorner, \mathrm{Sub}(k,k,k)\right)$	$\Gamma \vdash P\left(P(\xi)\right)$
	$\neg\, \underline{\mathrm{Pf}\left(\ulcorner \Gamma \urcorner, \mathrm{Sub}(k,k,k)\right)} \equiv \Gamma \nvdash P\left(P(\xi)\right)$
$m = \ulcorner Q \urcorner$	$Q(\xi) \equiv (\forall \bar{x})\left(\neg\, \underline{\mathrm{Pf}\left(\bar{x}, \mathrm{Sub}(k,k,k)\right)}\right) \equiv \nvdash P\left(P(\xi)\right)$

[3] In a non-standard model we add entities that are not numbers to the domain of numbers. For a description see Boolos and Jeffrey [1980] Chapter 17.

In the above table of correspondences we observe that $P \equiv \nvdash X(X(\xi))$, hence: -

$P(P) \equiv \nvdash P(P(\xi))$

On substituting $Q \equiv \nvdash P(P(\xi))$ into this we obtain: $Q \equiv \nvdash Q$.

Since $m = \ulcorner Q \urcorner$ then *m* is the Gödel number of a formula that is not a proof of itself.[4]

1.13 Lemma

If *K* is a sufficiently strong, ω-consistent, first-order theory, then: -

$\nvdash_K Q$ nor $\nvdash_K \neg Q$.

<u>Proof</u>

Let *K* be a sufficiently strong, ω-consistent, first-order theory. Then the sentence $Q \equiv (\forall \ulcorner \Gamma \urcorner)(\neg \mathrm{Pf}(\ulcorner \Gamma \urcorner, \mathrm{Sub}(k,k,k)))$

is a *definable* wff [see chapter 11] in *K*.

(1) $\nvdash_K Q$

The proof of this part, which does not depend on ω-consistency, is by contradiction.

1. $\vdash_K Q$ Assumption

Then, there exists a Γ, $\Gamma \vdash_K Q$.

So, there exists an $x \in \mathbb{N}$ such that x is the Godel number of a proof-sequence of Q

Let *n* be this number; that is $n = \ulcorner \Gamma \urcorner$. Then $\mathrm{Pf}(n, \ulcorner Q \urcorner)$.

Then $\underline{\mathrm{Pf}(n, \ulcorner Q \urcorner)} \equiv \Gamma \vdash_K Q$. Hence

2. $\vdash_K (\exists \bar{x}) \underline{\mathrm{Pf}(x, \ulcorner Q \urcorner)}$

Letting $m = \ulcorner Q \urcorner$, this gives $\vdash_K (\exists \bar{x}) \underline{\mathrm{Pf}(x, m)}$

3. $\vdash_K \neg(\forall \bar{x}) \neg \underline{\mathrm{Pf}(x, m)}$ $(\exists \bar{x})\phi \equiv \neg(\forall \bar{x})\neg\phi$

4. $\vdash_K \neg Q$ $Q \equiv (\forall \bar{x}) \neg \underline{\mathrm{Pf}(x, m)}$

5. $\nvdash_K Q$ $\vdash_K Q \wedge \neg Q$ is a contradiction

---

[4] The function $X(X(\xi))$ is known as the *norm* in the literature. For example, the various treatments by Smullyan [1992 and 1994] Let T_K denote the set of Gödel numbers of theorems of *K*. Let this be representable by an recursively enumerable (RE) formula $A(x)$. Let **u** be the Gödel number of $A(x)$. i.e. $\mathbf{u} = \ulcorner A(x) \urcorner$. Let $\mathrm{D}(\mathbf{u})$ be a function such that, if **u** is the Gödel number of a wff $A(x)$ with free variable *x*, then $\mathrm{D}(\mathbf{u})$ is the Gödel number of $A(\bar{\mathbf{u}})$. It can be shown that $\mathrm{D}(\mathbf{u})$ is primitive recursive. Then $\mathrm{D}(\ulcorner X(\xi) \urcorner) = \ulcorner X(X(\xi)) \urcorner$. It is equivalent to $\mathrm{Sub}(\ulcorner X \urcorner, \ulcorner X \urcorner, \ulcorner X \urcorner)$ in the previous arguments. Smullyan demonstrates the following result: If *K* is consistent and $\mathrm{D}(\mathbf{u})$ is representable in *K* then T_K is not expressible in *K*.

It is useful to explain the expression $\vdash_K (\exists \bar{x})\underline{\text{Pf}(x,\ulcorner Q \urcorner)}$ by showing how it would be decoded. Starting with $\vdash_K (\exists \bar{x})\underline{\text{Pf}(x,\ulcorner Q \urcorner)}$ we find the $\bar{n}$ which is the numeral such that $\underline{\text{Pf}(x,\ulcorner Q \urcorner)}$ is true. We map this back to the unique number *n* that it denotes. Then $n = \ulcorner \Gamma \urcorner$ for some proof sequence Γ of *Q*. So this gives us $\Gamma \vdash Q$. Note that the statement $\vdash_K Q$ means that *Q* follows from no *contingent* premises. But we expect to derive *Q* from the axioms of the theory *K* by using the rules of inference, so $\vdash_K Q$ means $\Gamma \vdash_K Q$ where Γ is a proof sequence *from the axioms*. The second part of the proof requires ω-consistency.

(2)	$\nvdash_K \neg Q$	
1.	$\vdash_K \neg Q$	
2.	$\nvdash Q$	by the consistency of *K*
3.	For every *n*, $\vdash_K \neg\underline{\text{Pf}(n,\ulcorner Q \urcorner)}$	
4.	$\nvdash_K (\exists \bar{x})\underline{\text{Pf}(x,m)}$	by ω – consistency and $m = \ulcorner Q \urcorner$
5.	$\nvdash_K \neg(\forall \bar{x})\neg\underline{\text{Pf}(x,m)}$	$(\exists \bar{x})\phi \equiv \neg(\forall \bar{x})\neg\phi$
6.	$\nvdash_K \neg Q$	
7.	$\vdash_K \neg Q$ and $\nvdash_K \neg Q$	
8.	$\nvdash_K \neg Q$	by reductio

1.14 Proof of Gödel's first incompleteness theorem

K is incomplete

Proof

Consider the expression

$(\forall x)\big(\neg\ \text{Pf}\big(x,\text{Sub}(\ulcorner Y \urcorner,\ulcorner Y \urcorner,\ulcorner Y \urcorner)\big)$ is not provable$\big)$

We have just supplied an instance of $\text{Sub}(\ulcorner Y \urcorner,\ulcorner Y \urcorner,\ulcorner Y \urcorner)$ which cannot be either proved or disproved. Hence this statement is valid. Specifically, we have: $(\forall x)\neg\text{Pf}(x,m)$ is not provable where $m = \ulcorner Q \urcorner$. Therefore, *Q* is valid but not provable. That is, we have $\vDash_K Q$ but not $\vdash_K Q$. Hence, *K* is incomplete.

1.15 Is the meta-language a conservative extension of the object-language?

The "location" of this proof is a matter of considerable debate in the literature. The proof demonstrates $\vDash_K Q$ and does so on the basis of a finite proof path; but the question is – does this proof take place in the object-language or the meta-language? Regarding this specific question, which is controversial, it is fortunately not necessary to decide, since we shall soon see that there is a generalised version of this theorem that *could not possibly be proven an any formal, analytic object-language whatsoever*, so the precise status of this proof is not critical to the debate.

However, I shall express an opinion. Clearly this proof does not take place in the object-language itself, and the real question is whether it takes place in a meta-language that is a isomorphic copy or at most an isomorphic copy of a recursive extension of the object-language – a meta-language that is itself also recursively axiomatic. The question is whether the meta-language ever requires principles that "go beyond" those of the analytic, formal object-language. For *circularity is the very essence of formalism* [Chap. 1 Sec. 7] Formalists can only ever "justify" any mathematical statement by reference to more formalism, so they shall never regard themselves as refuted unless it can be formally demonstrated that there is a meta-language that cannot be embedded within the object language or be regarded as a recursive extension of it. The argument above does not demonstrate this, and to do so requires a further proof. [See Section 4 below and Chapter 11]

2 The Gödel-Rosser theorem

The conclusion of Gödel's theorem is dependent on ω-consistency. It is, however, possible to construct another sentence, that we shall denote Q', that is undecidable in a sufficiently strong theory, *K*, assuming only that *K* is consistent.

2.1 Consistency property

For any wff ϕ, $\vdash \phi \Rightarrow \nvdash \neg\phi$.

(If *K* is complete, this can be strengthened to a biconditional, $\vdash \phi \Leftrightarrow \nvdash \neg\phi$.)

2.2 Outline of proof

Recall that $\mathrm{Pf}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)$ means $\ulcorner\Gamma\urcorner$ is the Gödel number of a sequence Γ proving the formula X with Gödel number $\ulcorner X\urcorner$. We now define $\overline{\mathrm{Pf}}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)$ to mean $\ulcorner\Gamma\urcorner$ is the Gödel number of a proof of the formula $\neg X$, where X is the formula with Gödel number $\ulcorner X\urcorner$. So $\underline{\overline{\mathrm{Pf}}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)} \equiv \Gamma \vdash \neg X$. We then define a particular sentence: -

$$P' \equiv \left(\forall \overline{\ulcorner\Gamma\urcorner}\right)\underline{\mathrm{Pf}\left(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner\right)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H}\urcorner} \le \overline{\ulcorner\Gamma\urcorner}\right)\underline{\overline{\mathrm{Pf}}\left(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner\right)}$$

This involves several backwards and forwards moves between the object- and meta-languages, so will need to pay careful attention to how it is constructed below. We let $k = \ulcorner P'\urcorner$ - that is k is the Gödel number of P'. Then substitute k for $\ulcorner X\urcorner$ in P' to obtain: -

$$m = \ulcorner Q'\urcorner \qquad Q' \equiv \left(\forall \overline{\ulcorner\Gamma\urcorner}\right)\underline{\mathrm{Pf}\left(\ulcorner\Gamma\urcorner,k\right)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H}\urcorner} \le \overline{\ulcorner\Gamma\urcorner}\right)\underline{\overline{\mathrm{Pf}}\left(\ulcorner\Gamma\urcorner,k\right)}$$

and let $m = \ulcorner Q'\urcorner$. We claim, provided the theory *K* is consistent, that m is the Gödel number of a formula that is not a proof of itself.

2.3 Demonstration

We once again begin by working through a table of correspondences that clarify the meaning of the statements P' and Q'.

Godel number, $\ulcorner \phi \urcorner$ / meta-language	Formula, ϕ / object-language
$n, k, \ldots$	$\bar{n}, \bar{k}, \ldots$ (numerals denoting numbers)
$x, y, \ldots$	$\bar{x}, \bar{y}, \ldots$ (variables for corresponding numerals)
$\ulcorner X \urcorner$	$X = X(\xi)$
$\ulcorner \Gamma \urcorner, \ulcorner \mathrm{H} \urcorner$	Γ, H represent proof sequences
$\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)$	$\Gamma \vdash X$
$\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)$	$\Gamma \vdash \neg X$
	$\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \equiv \Gamma \vdash \neg X$
	$\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)}$
	This is equivalent to $\Gamma \vdash X \supset \left(\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \mathrm{H} \vdash \neg X$
	If Γ proves X, then there exists a shorter proof of $\neg X$
$k = \ulcorner P' \urcorner$	$P' \equiv \left(\forall \overline{\ulcorner \Gamma \urcorner}\right) \underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)}$
	If there is any proof of X, then there exists a shorter proof of $\neg X$
$m = \ulcorner Q' \urcorner$	$Q' \equiv \left(\forall \overline{\ulcorner \Gamma \urcorner}\right) \underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, k)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, k)}$
	If there is any proof of P, then there exists a shorter proof of $\neg P$

This clarifies the meaning of the formulae involved.[5] As in the proof of Gödel's theorem, we prove this in two parts: -

(1) $\nvdash_K Q'$

The proof of this also falls into two parts. We construct one argument to show $\nvdash R$, where R is a statement, and then a second argument to show $\nvdash \neg R$.

Let $P' \equiv \left(\forall \overline{\ulcorner \Gamma \urcorner}\right)\left(\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \supset \left(\exists \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \underline{\overline{\mathrm{Pf}(\ulcorner \mathrm{H} \urcorner, \ulcorner X \urcorner)}}\right)$ and

$Q' \equiv \left(\forall \overline{\ulcorner \Gamma \urcorner}\right) \underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, k)} \supset \left(\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner}\right) \underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, k)}$

Suppose $\vdash_K Q'$. Then

[5] It is an arithmetised version of Grelling's paradox. [See also Chap. 14]

I 1. $\vdash_K Q'$

$\ulcorner Q' \urcorner$ is the Godel number of Q.

2. $\vdash_K (\exists \bar{x})\,\mathrm{Pf}(x, \ulcorner Q' \urcorner)$

Let j be the Godel number of a sequence proving Q'.

3. $\vdash_K \mathrm{Pf}(j, \ulcorner Q' \urcorner)$

Substituting $\ulcorner Q' \urcorner$ for $\ulcorner X \urcorner$ in

$P' \equiv (\forall \overline{\ulcorner \Gamma \urcorner})\,\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \supset (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)}$

we obtain

$(\forall \overline{\ulcorner \Gamma \urcorner})\,\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner Q' \urcorner)} \supset (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner Q' \urcorner)}$

Whence

4. $\vdash_K \mathrm{Pf}(j, \ulcorner Q' \urcorner) \supset (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \bar{j})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner Q' \urcorner)}$

Then by detachment

5. $\vdash_K (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \bar{j})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner Q' \urcorner)}$

Let $R \equiv (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \bar{j})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, \ulcorner Q' \urcorner)}$. So we have $\nvdash_K R$

II 1. $\vdash_K Q'$

The consistency property is $\vdash \phi \ \Rightarrow\ \nvdash \neg\phi$. Hence

2. $\nvdash_K \neg Q'$

This means that for every natural number n, $\overline{\mathrm{Pf}}(n, \ulcorner Q' \urcorner)$

is false. Recall that $\underline{\overline{\mathrm{Pf}}(n, \ulcorner Q' \urcorner)} \equiv H \vdash \neg Q$ where $n = \ulcorner H \urcorner$. Hence,

3. For every numeral $\bar{n}$, $(\vdash_K \neg \underline{\mathrm{Pf}(n, \ulcorner Q' \urcorner)})$.

As in (1) let j be the Godel number of a sequence proving Q'. Then

4. $\vdash_K \neg \underline{\mathrm{Pf}(0, \ulcorner Q' \urcorner)}$ and $\nvdash_K \neg \mathrm{Pf}\underline{(1, \ulcorner Q' \urcorner)}$ and ... and $\nvdash_K \neg \underline{\mathrm{Pf}(j, \ulcorner Q' \urcorner)}$

5. $\vdash_K \neg \underline{\mathrm{Pf}(0, \ulcorner Q' \urcorner)} \wedge \neg \underline{\mathrm{Pf}(1, \ulcorner Q' \urcorner)} \wedge \ldots \wedge \neg \underline{\mathrm{Pf}(j, \ulcorner Q' \urcorner)}$

6. $\vdash_K (\forall \overline{\ulcorner \mathrm{H} \urcorner} \le j) \neg \overline{\mathrm{Pf}}(\ulcorner \mathrm{H} \urcorner, \ulcorner Q' \urcorner)$

Quantifier logic has $(\forall x \le y) \neg\phi \Rightarrow \neg(\exists x \le y)\,\phi$. Hence

7. $\vdash_K \neg(\exists \overline{\ulcorner \mathrm{H} \urcorner} \le j)\,\overline{\mathrm{Pf}}(\ulcorner \mathrm{H} \urcorner, \ulcorner Q' \urcorner)$

That is

$\vdash_K \neg R$

The supposition $\vdash_K Q'$ leads to the contradiction $\vdash_K R \wedge \neg R$, whence $\vdash_K Q'$.

(2) $\nvdash_K \neg Q$

Suppose $\vdash_K \neg Q$. Then

1 $\vdash_K \neg Q'$

That is, $\vdash_K \neg(\forall \overline{\ulcorner \Gamma \urcorner})\,\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, k)} \supset (\exists\ \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner})\,\underline{\overline{\mathrm{Pf}}(\ulcorner \Gamma \urcorner, k)}$

Where $k = \ulcorner P' \urcorner$ and $P' \equiv (\forall \overline{\ulcorner \Gamma \urcorner})\left(\underline{\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)} \supset (\exists \overline{\ulcorner \mathrm{H} \urcorner} \le \overline{\ulcorner \Gamma \urcorner})\,\underline{\overline{\mathrm{Pf}(\ulcorner \mathrm{H} \urcorner, \ulcorner X \urcorner)}}\right)$

Let j be the Godel number of a sequence proving $\neg Q'$

2. $\vdash_K \underline{\overline{\mathrm{Pf}}(j, \ulcorner Q' \urcorner)}$

The consistency property is $\vdash \neg\phi \ \Rightarrow\ \nvdash \phi$. Hence

3. $\nvdash_K \underline{\mathrm{Pf}\left(j,\ulcorner Q'\urcorner\right)}$

So $\mathrm{Pf}\left(n,\ulcorner Q'\urcorner\right)$ is false for all natural numbers n. Hence

4. For every natural number n, $\vdash_K \neg\underline{\mathrm{Pf}\left(n,\ulcorner Q'\urcorner\right)}$

5. $\vdash_K \neg\underline{\mathrm{Pf}\left(0,\ulcorner Q'\urcorner\right)}$ and $\nvdash_K \neg\underline{\mathrm{Pf}\left(1,\ulcorner Q'\urcorner\right)}$ and ... and $\nvdash_K \neg\underline{\mathrm{Pf}\left(j,\ulcorner Q'\urcorner\right)}$

6. $\vdash_K \neg\mathrm{Pf}\underline{\left(0,\ulcorner Q'\urcorner\right)} \wedge \neg\mathrm{Pf}\underline{\left(1,\ulcorner Q'\urcorner\right)} \wedge \ldots \wedge \neg\mathrm{Pf}\underline{\left(j,\ulcorner Q'\urcorner\right)}$

7. $\vdash_K \left(\bar{y} \le \bar{j}\right) \supset \neg\underline{\mathrm{Pf}\left(y,\ulcorner Q'\urcorner\right)}$

Now consider

A. $\vdash_K \bar{j} \le \bar{y}$ by hypothesis

B. $\vdash_K \underline{\overline{\mathrm{Pf}|}\left(j,\ulcorner Q'\urcorner\right)}$ from line 2

C. $\vdash_K \left(\bar{j} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(j,\ulcorner Q'\urcorner\right)}$ by $A, B, \wedge - I$

D. $\vdash_K (\exists\bar{z})\left(\left(\bar{z} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(z,\ulcorner Q'\urcorner\right)}\right)$ from C by $\exists - I$

Then from $A - D$ by the deduction theorem, we have

8. $\vdash_K \bar{j} \le \bar{y} \supset (\exists\bar{z})\left(\left(\bar{z} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(z,\ulcorner Q'\urcorner\right)}\right)$

The following line states a theorem of any sufficiently strong theory K

9. $\vdash_K \left(\bar{y} \le \bar{j}\right) \vee \left(\bar{j} \le \bar{y}\right)$

Combining line 9 with the results at 7 and 8, we have

10. $\vdash_K \neg\underline{\mathrm{Pf}\left(y,\ulcorner Q'\urcorner\right)} \vee (\exists\bar{z})\left(\left(\bar{z} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(z,\ulcorner Q'\urcorner\right)}\right)$

By the tautology $\vdash \neg p \vee q \equiv p \supset q$

11. $\vdash_K \underline{\mathrm{Pf}\left(y,\ulcorner Q'\urcorner\right)} \supset (\exists\bar{z})\left(\left(\bar{z} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(z,\ulcorner Q'\urcorner\right)}\right)$

By generalisaton

12. $\vdash_K (\forall\bar{y})\underline{\mathrm{Pf}\left(y,\ulcorner Q'\urcorner\right)} \supset (\exists\bar{z})\left(\left(\bar{z} \le \bar{y}\right) \wedge \underline{\overline{\mathrm{Pf}|}\left(z,\ulcorner Q'\urcorner\right)}\right)$

This just is $\vdash_K Q'$. We have a contradiction. Thus

13. $\vdash_K \neg Q$

2.4 Gödel-Rosser first incompleteness theorem

As in the first version of this theorem: $\vDash_K Q$ but not $\vdash_K Q$. Hence, K is incomplete.

2.5 Corollary

Let K be any consistent, sufficiently strong theory. Then every consistent recursively axiomatisable extension of K is subject to the Gödel-Rosser Theorem, and therefore has an undecidable sentence. That is, K is essentially incomplete.

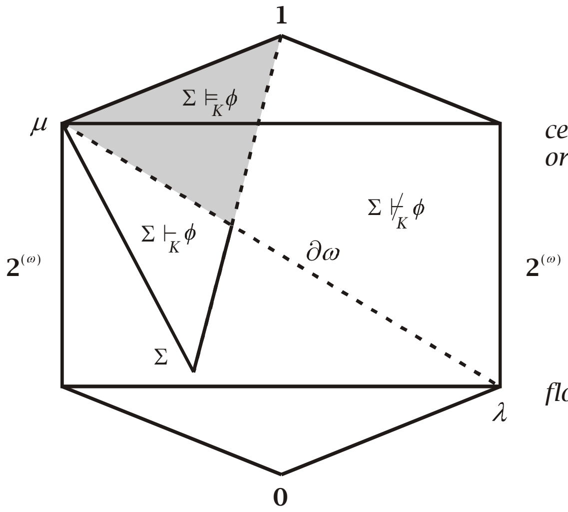
3 Analysis and interpretation of Gödel's theorem

Let *K* be any consistent, sufficiently strong theory. *K* is a formal analytic logic and therefore has a model; this model is a lattice, *L*, and *K* is the logic that is built over this lattice [see 7/2.2 for justification of the assumption that the model is a lattice]. *K* is not complete; hence the lattice *L* must differ in some significant way from any lattice in which a complete theory is realised. We have seen that the maximal lattice of a complete theory is $CF(\omega)$, which is the Boolean algebra of the finite and co-finite subsets of ω. $CF(\omega)$ is a proper subset of $\mathbf{2}^{\omega}$, the Cantor set, and the difference between the two, $\mathbf{2}^{\omega} - CF(\omega)$ constitutes the *boundary* between the two parts of $CF(\omega)$, which are isomorphic copies of the prime ideal of all finite subsets of ω; we write $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$. In terms of the *Klein bottle model* [6/4.9] of the Cantor set, the two parts of $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$ correspond to the two sides of the Klein bottle with boundary $\mathbf{2}^{\omega} - CF(\omega)$. It is a feature of this model that while both sides of the model can be accessed from each other by paths lying wholly on the surface, which is globally one-sided and non-orientable, those paths are non-compact and infinite, being equivalent to a path of ordinal length ω. A sufficiently strong theory adds the boundary to $CF(\omega)$ and transforms the model into $\mathbf{2}^{\omega}$, or at the least some minimal version of it that is consistent with the notion of a boundary.[6] A sufficiently strong theory embeds a principle of complete inductive reasoning. However, the partition on which the analytic logic is based, namely the subdivision of an interval $[0,1]$ into at least ω actual parts, results in notional atoms represented by the individual singleton subsets, $\{0\}, \{1\}, \{2\}, \ldots$, but these are not ordered, for if so they were they would generate a chain rather than a lattice. The property of sufficient strength imposes upon this structure an additional external structure of the well-ordering of the natural numbers. This is encapsulated in the model by the presence in the one-point compactification $\mathbb{N}_{\infty} = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda$ of the chain $\lambda = \{\mathbb{N}\}$, where μ and $\mathbb{N}_{\infty}$ represent the antichain over which the analytic logic is constructed.

The embedded inductive procedure provides a two-step process that enables one to *pass directly to the boundary*, which is equivalent in the Klein bottle model to attaining the boundary by passing through it to a generic set. Induction in the meta-language enables one to show that there are generic, "non-constructive" yet definable sets lying in the boundary [See Chap. 11 for a discussion of definability]. Among these are objects corresponding to singleton sets of transcendental numbers; the solution to the Halting problem is also a non-constructible generic set. So induction as a rule of reasoning transcends the domain of the

[6] The exact determination of the minimum model of a sufficiently strong theory is likely to depend on the solution to the continuum hypothesis.

effectively computable analytic logic built over ω, which is isomorphic at most to $\mathbf{2}^{<\omega}$. Let K represent the axioms of any sufficiently strong language, *K*; let *L* be the lattice corresponding to *K*; let Σ represent the least lattice point of a filter in *L*. Note, we think of *L* as the Cantor set, $\mathbf{2}^{\omega}$. Then Gödel's incompleteness theorem corresponds to the following diagram of $L \cong \mathbf{2}^{\omega}$: -



Here the shaded part corresponds to the region of the lattice proven to exist by Gödel's theorem where we have $\Sigma \vDash \phi$ but $\Sigma \nvdash \phi$. This diagram is already an intuitive proof of Poincaré's thesis – for we "see" immediately that the boundary is an inalienable feature of the Cantor set and that it could never be either eradicated or imported into the domain of the effectively computable, which is at most $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$ or an isomorphic copy thereof. Parts of the boundary could be mapped into $CF(\omega) = \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$ but never all of it: the boundary is inexhaustible. Gödel's theorem therefore establishes the synthetic conclusion that there exists a boundary within the lattice corresponding to any sufficiently strong formal, analytic lattice. However, formalists will never agree until this result is formally established: this can be achieved by several different though related methods.

4 Proof of Poincaré's thesis from Gödel's theorem by induction in the meta-language

The counter-argument to the possibility of such a proof of Poincaré's thesis is stated by J.J.C. Smart: -

> Suppose that a machine, by observing its own linguistic behaviour could ascertain its own syntax ... There is pretty certainly no *a priori* argument of the Rosenbloom or similar sort against the possibility of such a machine. If such a machine were

> able to ascertain its own syntax it could presumably, if it had enough storage units, progressively keep on adjoining new syntax languages to its own programme, converting itself from an L_0 machine to an L_1 machine to an L_2 machine, and so on indefinitely…" (Smart [1961])

He cites Quine as an authority and states that Quine has a purely mathematical rebuttal: -

> He [Quine] has pointed out to me that a machine which was programmed with a set of axioms for elementary number theory could be modified by adding on as an additional axiom the corresponding Gödelian undecidable sentence, or the arithmeticised form of the sentence, to the effect that the axioms and rules are consistent. This process could be iterated and the general method for constructing the Gödel sentence at any stage could itself be mechanised so as to produce an axiomatisation of elementary number theory. This process could be continued through all the constructible ordinals. There would nevertheless always be a Gödelian undecidable sentence. (Smart [1961] – from the concluding paragraph.)

My underlining. The possibility that Smart refers to here is as follows. Let K_0 be a sufficiently strong first-order theory. Then there exists a formula Q_0 of K_0 such that $\vDash_{K_0} Q_0 \wedge \nvdash_{K_0} Q_0$. Then $K_1 = K_0 + Q_0$ is consistent and in K_1 we have $\vdash_{K_1} Q_0$. The claim is that we can generate K_1 recursively from K_0. This latter claim is made on the assumption that the proof of Gödel's theorem we have given above takes place entirely in the meta-language which is arithmetic and first order; that all the objects generated are done so recursively, and so forth. This is probably valid. Hence it is also valid that we can recursively generate K_1 from K_0, K_{n+1} from K_n. Smart claims, "This process could be continued through all the constructible ordinals". Actually, the meta-language, being arithmetic, does not have constructible ordinals, and the induction appropriate there is complete induction over the natural numbers, $\mathbb{N}$, which is appropriate, because we would like to keep the mathematics in the meta-language out of the actual infinite – this being one interpretation of "finitism" [1.2.1 and 14.2.3]. Nonetheless, even granting Smart transfinite induction over all constructible ordinals, that is not sufficient to establish the principle that the proof of Gödel's theorem is recursive (effectively computable). This Smart acknowledges in his concluding words from the passage underlined above, "There would nevertheless always be a Gödelian undecidable sentence." This is equivalent to the observation on the diagram about *that there will always be a boundary in the Cantor set.* The form that Smart takes Gödel's theorem *explicitly* in his paper is: -

4.1 (+) One step Gödel's theorem

Let K_0 be a sufficiently strong logic. Then $(\exists X)\left(\nvdash_{K_0} X \text{ and } \vDash_{K_0} X\right)$

But *this is not Gödel's theorem* – as indeed the argument from Smart's own paper implicitly indicates. That is why I have called it the *one step Gödel's theorem*. Gödel's theorem is a proof by induction in the meta-langauge that

4.2 (+) Universal Gödel's theorem

Let K be any sufficiently strong logic. Then: -

$$G \qquad (\forall K)(\exists X)(\vDash_K X \wedge \nvdash_K X)$$

G is asserted absolutely without restriction to class or set. We have a proof in the meta-language of this assertion. We could write $\vdash_\Omega G$, where Ω denotes the metalanguage. I shall show that the relation $\vdash_\Omega$ is *not in the object language*, where the relation is denoted $\vdash_K$.

4.3 Theorem

G is not a deductive consequence of any formal analytic logic.

<u>Proof</u>

Now suppose there exists a recursive language K^* (with corresponding lattice L^*) such that $(\exists K^*)(\vdash_{K^*} G)$. In other words, we try to embed the proof of G into a compact domain with finite proof paths. Then: -

$\vdash_{K^*} G$	
$\vdash_{K^*} (\forall K)(\exists X)(\vDash_K X \wedge \nvdash_K X)$	
$\vdash_{K^*} (\exists X)(\vDash_{K^*} X \wedge \nvdash_{K^*} X)$	Substituting $K^* = K$.
$\vdash_{K^*} (\vDash_{K^*} Q \wedge \nvdash_{K^*} Q)$	Letting $X = Q$ were Q is the specific Godel sentence
$\vdash_{K^*}\vDash_{K^*} Q \quad \wedge \vdash_{K^*}\nvdash_{K^*} Q$	
$\vdash_{K^*} Q \quad \wedge \nvdash_{K^*} Q$	$(**)$

This is a contradiction. Therefore, $\nvdash_{K^*} G$. The line at $(**)$ uses two rules in the metalanguage that are intuitively self-evident and can be further clarified as follows: -

1. Let Y be a proposition of a lattice K, then

   Y is true of $K \to \vDash_K Y$ for all filters Σ in K

   The statement Y is in the meta-language. It implies no specific relation between points in a lattice and is therefore asserted *absolutely* of all lattices. If any statement Y is asserted absolutely in the meta-language then it necessarily implies that Y is true every lattice and true for all filters whatsoever. Every lattice and filter whatsoever is a model of Y. This rule is abbreviated to: $Y \to \vDash Y$

I shall call this principle *concatentation of consequence*, since it allows the symbol to be attached in front of any other symbol asserted absolutely or in the form $\vDash Y$. From this rule it follows that $\vdash \vDash Y$ iff $\vdash Y$.

2. Suppose $\vdash \nvdash X$. Then

$\vdash \neg \vdash X$	Definition $(\nvdash X \equiv \neg \vdash X)$
$\neg \vdash \vdash X$	Consistency $(\vdash \neg Y$ and $\vdash Y$ are inconsistent. Here $Y = \vdash X)$
$\neg \vdash X$	Rule for proofs $(\vdash \vdash X$ iff $\vdash X)$
$\nvdash X$	

Formalists take the view that any given axiom is recursive. Let *A* be an axiom; then we $A \vdash A$, which seems to confirm their point. But suppose we attempt to adopt *G* as an axiom and write $G \vdash_G G$. The Universal Gödel theorem, applied to itself, yields the conclusion $G \vdash_G G \Rightarrow G \nvdash_G G$.[7] $G \nvdash_G G$ says that there is a path in a lattice *G* from *G* to *G*. As the path is simply the point *G* what this shows is that *G* cannot be a point in *any lattice whatsover*. The metalogic in which there is a proof of *G* is *wholly different* from any analytic logic. It is not a relation in a lattice.

4.4 Theorem

There exists a synthetic logic.

Proof

G is true for all lattices that correspond to sufficiently strong languages, *K*. So we have $\vDash_K G$. Together with the statement, $\nvdash_K G$ this yields: -

$$F \qquad (\forall K)(\vDash_K G \text{ and } \nvdash_K G)$$

So *G* and *F* are statements that are true but are not analytic proofs of any lattice whatsoever. In the above argument the statements *G* and *F* are necessarily true. Therefore, their contraries cannot be adopted as axioms. Therefore, *there is a synthetic logic.*

4.5 Observation

Suppose we have a sufficiently strong logic *K*, then there is a Gödel sentence *Q* for *K*. We are not *forced* to adopt *Q* as an axiom because there is still an element of *choice* in the matter.

[7] This is highly reminiscent of the Liar Paradox which is not surprising, since Gödel's theorem is explicitly constructed on the idea of arithmetising the Liar paradox. This invites further investigation of the Liar paradox itself, which follows below [Chapter 10],. However, the "paradox" can be resolved independently.

However, if we wish to ascend to a stronger theory in which Q can be proven, then we can only do that by appending Q as an axiom. It is not consistent to add $\neg Q$. So there is only one consistent extension of the theory. We ascend to $K+Q$ but in doing so generate a second Gödel sentence, and so on, *ad infinitum*.

4.6 Turing's investigation

We have to consider the possibility that Gödel's incompleteness result may be overcome by means of an ordinal logic, which was the suggestion that Smart, quoting Quine, also claimed could prevent the conclusion that there exists a synthetic logic. This idea was investigated by Alan Turing when he was set the problem by Alonzo Church as a Ph.D thesis at Princeton, where Turing was between 1936 and 1938. It was carried out in Church's λ-calculus.

Before examining Turing's conclusions, let us observe, as we did above [4.10.4 et seq] that it is always trivially possible to overcome the incompleteness result. When discussing the proof of the completeness of the predicate calculus, $\Sigma \vDash \phi \Rightarrow \Sigma \vdash \phi$, I observed that if one lengthens the proof paths so that $\Sigma \vDash \phi$ iff $\Sigma \vdash \phi$ then completeness becomes true by definition. So every lattice is trivially complete. Actually, the incompleteness result is a question of the relation between lattices and their embeddings in each other. To say that the lattice $CF(\omega) \cong \mathbf{2}^{<\omega} \cup \mathbf{2}^{<\omega}$ is incomplete means that it does not contain all its infinite meets and joins and can be embedded in a larger lattice that does. In a sufficiently strong logic K the presence of a form of complete induction permits the definition of generic sets that actually construct from within the logic the lattice points that do not belong to $CF(\omega)$; we can now appreciate that this is the product of synthetic logic, and it is clear that all completion arguments are aspects of synthetic logic, as I shall discuss further below. [15/4.2]

Nonetheless, there is a sense in which the Cantor set, $\mathbf{2}^{\omega}$, is trivially complete, both as a logic and as a topology. Yet this provides no comfort whatsoever to formalism or to advocates of Strong AI, because the proof paths from finite to generic sets are decidedly not finite or compact in the formal sense. Hence, they are not computably effective. Whenever one has to complete an actual infinity then there is a break down of effective computability. One has transcended the maximally effectively computable domain, $CF(\omega)$. Again, an expression like $\{e\}$ where e is the usual transcendental number, is grasped in consciousness as a finite object referring to an actual, completed infinite process, but cannot be generated mechanically as such. The whole philosophical problem arises from the philosophical error that assumes we must interpret denotation as given wholly in extension. *Meaning transcends extension.*

However, lest even now this be disputed, let us examine the conclusions of no lesser a person than Turing [1939] himself on this question, both in his own words and as reported by Feferman [1988]. In that paper, Turing introduces the idea as follows: -

> The well-known theorem of Gödel *1931* shows that every system of logic is in a certain sense incomplete, but at the same time it indicates means whereby from a system L of logic a more complete system L' may be obtained. By repeating the process we get a sequence $L, L_1 = L', L_2 = L_1', \ldots$ each more complete than the preceding. A logic L_ω may then be constructed in which the provable theorems are the totality of theorems provable with the help of the logics $L, L_1, L_2, \ldots$. Proceeding in this way we can associate a system of logic with any constructive ordinal. It may be asked whether such a sequence of logics of this kind is complete in the sense that to any problem A there corresponds an ordinal α such that A is solvable by means of the logic L_α.

He also states that it is trivial to show that there exist complete ordinal logics. The non-trivial case collapses on the problem of invariance: -

> *Invariance of ordinal logics.* An ordinal logic Λ is said to be *invariant up to* an ordinal α if, whenever Ω, Ω' are ordinal formulae representing the same ordinal less than α, the extent of $\Lambda(\Omega)$ is identical to the extent of $\Lambda(\Omega')$. An ordinal logic is *invariant* if it is invariant up to each ordinal represented by an ordinal formula." (Turing [1939] p.200)
>
> I shall prove that an ordinal logic Λ cannot be invariant and have the property that the extent of $\Lambda(\Omega)$ is a strictly increasing function of the ordinal represented by Ω." (Turing [1939] p.203.)

The ultimate conclusion is: -

> ... with almost any reasonable notation of ordinals, completeness is incompatible with invariance. (Turing [1939] p.209.)

Feferman describes the upshot thus: -

> The demand on intuition on recognizing "which formulae are ordinal formulae" is greater than Turing realised. The work "was subsequently done, at the suggestion of Kreisel *1958*, by restricting attention, successively, to those notations a for which one has a *proof* in L_b for some $b < Oa$ that $a \in O$ (i.e., that a represents a well-ordering). These have come to be called *autonomous ordinal notations*, and the notion of ordinal logic restricted in this way, *autonomous recursive progressions of axiomatic theories.* (Feferman [1988] p.131)

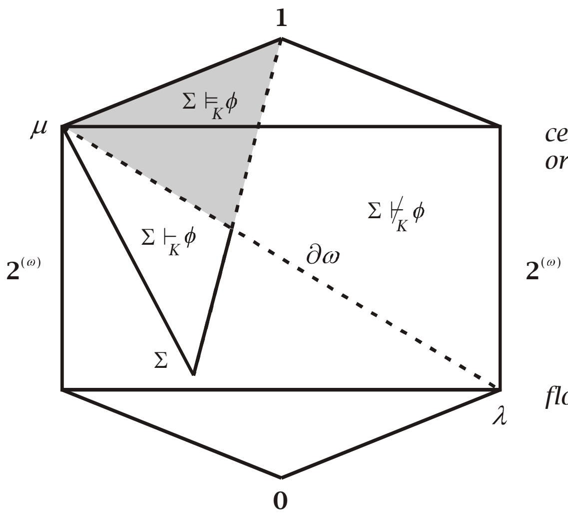
And in conclusion: -

> It is easily shown that the $\bigcup L_a$ [*a* autonomous] is recursively axiomatizable and hence incomplete by Gödel's theorem." (Feferman [1988] p.131)

So ordinal logics do not solve the problem of incompleteness.

5 Proof of Poincaré's thesis from Gödel's theorem by consideration of generic sequences

I refer again to the following diagram: -

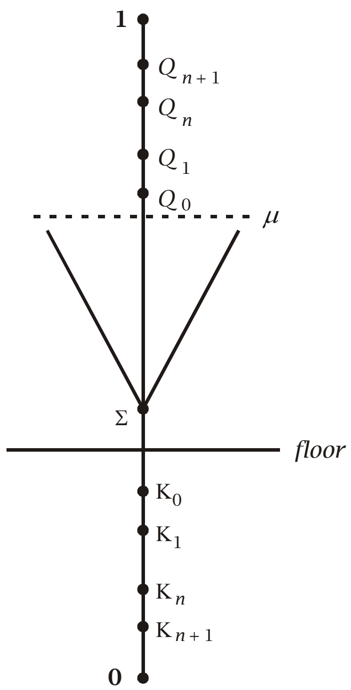


Considering the Gödel sentence, *Q*, for which we have, $\vDash_K Q$ but $\nvdash_K Q$, the question I wish to consider is: *where in the lattice does Q appear?* At first glance $\vDash_K Q$ affirms that *Q* is true of every filter whatsoever, and hence must be a name of **1**, that is a member of the equivalence class $\mathrm{Eq}(\mathbf{1})$. Members of this class are joins of **1**, where each **1** is equivalent to the law of excluded middle. But *Q* cannot be recursively generated in the list of joins, so must represent a non-recursive infinite list of **1**s. This is intriguing because it shows that the elements of $\mathrm{Eq}(\mathbf{1})$ cannot be recursively generated in their entirety, yet we know and can prove that *Q* is a member of this class. This points strongly to the existence of a *logic of intensions* that enables us to establish the extensional equivalence of concepts even where no mechanical process for this could possibly exist.

Reflecting momentarily on the finite case, consider yet again the finite Boolean algebra $\mathbf{2}^4$ based on a partition into four disjoint atoms $\alpha_1 \equiv \{1\}, \alpha_2 \equiv \{2\}, \alpha_3 \equiv \{3\}, \alpha_4 \equiv \{4\}$. Let

Ax_2 denote the axioms of the propositional caculus. These may be conjoined to any lattice point; for example, $\alpha_1 \equiv \alpha_1 \wedge \mathrm{Ax}_2$. The axioms are true of all filters whatsoever, hence true in $\mathrm{filter}(\{1\})$, $\mathrm{filter}(\{2\})$, and so forth; hence true in the *meets* of all these filters: $\{1\} \wedge \{2\} \wedge \{3\} \wedge \{4\}$. But in $\mathbf{2}^4$ this meet does not exist, save as the **0**. The expression means true in the intersection of the filters, which is in $\mathbf{1} \equiv \{1,2,3,4\}$, which is what we expect as the axioms Ax_2 are a name of **1**.

The Cantor set, $\mathbf{2}^\omega$ is generated by a partition of $[0,1]$ into $\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}\} = \mu \cup \lambda$. The partition of $\mu \cong \mathbb{N}$ into notional atoms $\{0\}, \{1\}, \{2\}, \ldots$ allows for the definition in the language K built over the lattice of their meet: $\{0\} \wedge \{1\} \wedge \{2\} \wedge \ldots$. This meet is not in the lattice $CF(\omega)$ but does define a generic element in the complete lattice in which it is embedded. This element is not identical to **0**, being not the meet of all the atoms (λ has been left out.) If it did equal **0** the lattice would collapse to **0** and the whole system would be inconsistent. The consistency of the system is maintained by the rule $\{0\} \wedge \{1\} \wedge \{2\} \wedge \ldots \neq \mathbf{0}$. Let $\mathrm{K}_0 \equiv \{0\} \wedge \{1\} \wedge \{2\} \wedge \ldots$ represent this meet. This places K_0 below the notional floor of the lattice. Let Σ represent an arbitrary filter in the set of all finite subsets of ω. Now we have $\mathrm{K}_0 \vDash_K Q_0$ and $\mathrm{K}_0 \nvdash_K Q_0$. This places Q_0 above the boundary between the finite subsets of ω, $F(\omega)$; it is a member of every filter of $F(\omega)$. Let $\mathrm{K}_1 = \mathrm{K}_0 + Q_0$; we have $\mathrm{K}_1 \vdash_K \mathrm{K}_0$ and $\mathrm{K}_1 \vdash_K Q_0$. This information can be represented diagrammatically: -



The effect of the omission of the atom λ from the ideal $F(\omega)$ is to make Q be seen from within $F(\omega)$ as lying above the boundary μ but in a position below **1**. Iteration of the construction $\mathrm{K}_{n+1} = \mathrm{K}_n + Q_n$ creates a generic sequence of lattice points, $\mathrm{K}_0, \mathrm{K}_1, \ldots$ all lying in the neighbourhood of **0** but not identical to it. The effect of each iteration of this process is to move each Q_n inside the upper boundary μ of the ideal; the boundary is never eliminated by this process.

Let $\mathrm{G} = \bigcup_n \mathrm{K}_n$, where the limit is taken over all $n \in \mathbb{N}$ or all transfinite ordinals (as desired). Then the assumption that G is encoded by some finite information, $\mathrm{G} = \mathrm{K}_n$ leads to a contradiction; hence G is a generic set: *G* is doubly transcendent generic set, because no complete recursive process can ever construct it – not even as an element of the Cantor space, $\mathbf{2}^{\omega}$. G can never be reached by any iterative process whatsoever, we conclude that $\mathrm{G} = (\omega)$, which is the boundary between the finite and infinite subsets of ω and can never be an element of any set whatsoever. The existence of *G* is demonstrated by the Universal Gödel theorem, which is a synthetic proof by mathematical induction in the meta-language. There is no possible way in which the existence of G could be demonstrated from within effective, formal, analytic logic; it is not possible to derive the existence of G from any mere partition of space.

6 Proof of Poincaré's thesis from Gödel's theorem by means of fixed points

A proof is a finite sequence of formulas from a starting point, one that is included in the sequence, to its terminal point. A proof corresponds to a finite sequence of points in a lattice. Proof paths proceed up a filter with each higher lattice point being a dilution of one below. Let $\alpha_i, \alpha_j, i \neq j$ be notional atoms: -

$\vdash \alpha_i$
$\vdash \alpha_i \vee \alpha_j$

Allowing $\vdash$ to be multiple premise and multiple conclusion, all premises can be combined into a single point *p* in the lattice by meet, and all conclusions belong then to the *finitely generated* filter of this point. Denote this finitely generated filter by $p\vdash$. Hence a proof from *p* is a compact path in $p\vdash$. Let Γ be a proof path in $p\vdash$, which means it must start at *p*. Then $\mathrm{Pf}(\ulcorner\Gamma\urcorner, \ulcorner X\urcorner)$ says, "the formula *X* with Gödel number $\ulcorner X\urcorner$ is a member of $p\vdash$". To illustrate further the relation $\ulcorner X\urcorner \in (X\vdash)$, first let us encode by Gödel numbers the *canonical* one-step proof: -

$\vdash \alpha_i$
$\vdash \alpha_i \vee \alpha_j$

This is the concatenated sequence: -

$\alpha_i \circ (\alpha_i \vee \alpha_j)$

Looking at one standard version of Gödel numbering in more detail, Gn first codes the separate formulas α_i and $\alpha_i \vee \alpha_j$ to numbers $n_i = \ulcorner \alpha_i \urcorner$ and $n_2 = \ulcorner \alpha_i \vee \alpha_j \urcorner$ and then the sequence $\alpha_i \circ (\alpha_i \vee \alpha_j)$ to $2^{\ulcorner \alpha_i \urcorner} \cdot 3^{\ulcorner \alpha_i \vee \alpha_j \urcorner}$. There is a homomorphism from the sequence of formulae $s_1 \circ s_2 \circ \ldots \circ s_n$ to the formula $s_1 \wedge s_2 \wedge \ldots \wedge s_n$; note here s_1 is the formula denoting the base of the filter, $s_1 \wedge s_2 \wedge \ldots \wedge s_n$ is the path in the filter and s_n is the conclusion – the formula that is proven from s_1. Note $t = s_1 \wedge s_2 \wedge \ldots \wedge s_n$ is a proof of *t* from *t*, equivalent to the law of identity $t \supset t$. The homomorphism

$$\phi \begin{cases} \text{proof paths} \to \text{formulae} \\ s_1 \circ s_2 \circ \ldots \circ s_n \to s_1 \wedge s_2 \wedge \ldots \wedge s_n \end{cases}$$

is a homomorphism of finite proof paths to lattice points. In other words: -

> **6.1 (+) Result**
> *Every finitely generated lattice point* is the name of a finite proof path and conversely.

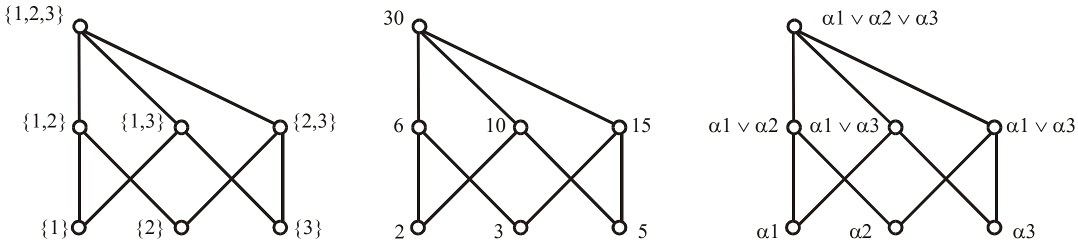
Let us show this in an alternative way. Given the encoding: -

$$\phi \begin{cases} \text{proof paths} \to \text{formulae} \\ s_1 \circ s_2 \circ \ldots \circ s_n \to s_1 \wedge s_2 \wedge \ldots \wedge s_n \end{cases}$$

Then our canonical one-step proof $\alpha_i \circ (\alpha_i \vee \alpha_j)$ becomes $\alpha_i \wedge (\alpha_i \vee \alpha_j) \equiv \alpha_i$. In other words every lattice point *p* is a name for its finitely generated filter $p \vdash$, and every proof path in this filter is logically equivalent to this name *p*. So the Gödel numbering is an encoding of all finite proof paths into the natural numbers. All finite proofs are finitely generated lattice points and Gödel number maps lattice points to numbers. Therefore the domain of $\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner)$ is $\mathbf{2}^{<\omega} \cong \mathbb{N}$ and under Gödel numbering: -

$$\mathrm{Pf}(\ulcorner \Gamma \urcorner, \ulcorner X \urcorner) : \mathbf{2}^{<\omega} \to \mathbb{N} \subset \mathbb{N}_\infty \cong \omega .$$

$\mathbf{2}^{<\omega}$ is an ideal in $\mathbf{2}^{\omega}$ and there are lattice points in $\mathbf{2}^{\omega}$ that correspond to actually infinite subsets of ω. The concept of Gödel numbering is essentially based upon the isomorphism of structures all represented by $\mathbf{2}^{<\omega}$.



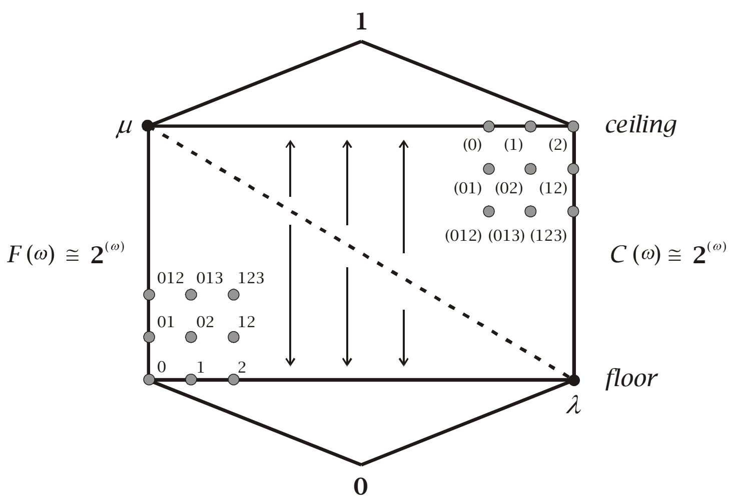
Power set Prime divisors Atomic propositions

Now observing that every proof path is logically equivalent to the minimum lattice point in which it lies, and contracting the collection of all proof paths to these lattice points, we see that every lattice point is the name of an equivalence class of compact proofs and every compact proof belongs to such an equivalence class. Hence, there is a one-one pairing of these equivalence classes and the natural numbers that is given by the diagonalisation argument.

Lattice points ≅ equivalence classes of compact proofs							Natural numbers ℕ					
{0}	{1}	{2}	{3}	{4}	...		1	2	6	7	...	...
{01}	{02}	{12}	{03}	{13}	...		3	5	8	...	...	...
{012}	{013}	{023}	{123}	{014}	...	⟷ (1–1)	4	9	...	...	...	...
{0123}	{0124}	{0134}	{0234}	{1234}	...		10	...	...	...	...	...
...	...	...	...	...	...		...	...	...	...	...	...

6.2 (+) Gödel numbering as a contraction mapping

So let us take the above mapping as our canonical representation of Gödel numbering. Then Gödel numbering is a *contraction mapping* of the ideal $\mathbf{2}^{<\omega}$ onto its floor of notional atoms. By symmetry (duality) we may extend this mapping to the filter of co-finite sets in $\mathbf{2}^{\omega}$ onto its ceiling of notional co-atoms (Boolean primes).



In this diagram the atoms $\{0\}, \{1\}, \{2\}, \ldots$ are represented by digits, 0, 1, 2, ...; and the co atoms $\omega - \{0\}, \omega - \{1\}, \omega - \{2\}, \ldots$ by $(0), (1), (2), \ldots$.

This contraction mapping represents re-labellings of the atoms: each lattice point originally in $F(\omega) \cong \mathbf{2}^{<\omega}$ becomes a new notional atom. Hence Gödel numbering in this sense is equivalent to the process earlier defined as *lowering the floor* of the lattice. [5/5.8 and 7/1.6] We see automatically that whenever the floor is lowered what results is just another lattice that is isomorphic to the original. The process of lowering the floor never enables one to transcend the lattice, here $\mathbf{2}^{<\omega}$.

6.3 (+) The effect of automorphisms of the Cantor set

The symmetry engendered by the duality of the inversion of the order relation of the lattice introduces into $\mathbf{2}^{\omega}$ a *line of symmetry*, that is its boundary. The sufficient strength of the language enables the definition of automorphisms of this space [Chap. 11 Sec. 2] which have fixed points or lines of symmetry. The simple reflection of $\mathbf{2}^{\omega}$ that maps $\Theta : F(\omega) \to C(\omega)$ and conversely, has a line of symmetry, which swaps all elements of the boundary as well, for we have $\Theta(\mu) = \lambda$. The line of symmetry is not a set in the lattice, but exists nonetheless. Since it does not exist in the lattice its existence could never be demonstrated by analytic inference within a filter of the lattice; it is not a formal consequence of analytic logic, though it is a property of one, provided it has sufficient strength; the existence of the symmetry is demonstrated by synthetic logic, here derived from geometric intuition and analogy.

6.4 (+) The extended Gödel map

The question is: can we extend this form of Gödel numbering given above [6.2 above] to the whole lattice, $\mathbf{2}^{\omega}$? The ideal $F(\omega) \cong \mathbf{2}^{<\omega}$ and filter $C(\omega) \cong \mathbf{2}^{<\omega}$ are both equinumerous to $\aleph_0$, so the bulk of $\mathbf{2}^{\omega}$ is contained in the boundary, here shown by the line joining the atom λ to the co-atom μ. The boundary is a huge set of cardinality $2^{\aleph_0}$. If such an extension of Gödel numbering to the whole of $\mathbf{2}^{\omega}$ is possible, the contraction of the compact part $CF(\omega)$ to the floor and ceiling result in a dilation of the boundary as it expands to fill the "space" left by the contraction. This is possible because infinity is inexhaustible.

Proof paths in $F(\omega) \cong \mathbf{2}^{<\omega}$ are compact, and hence bounded above by the indeterminate (ω); when we enter the boundary region we find lattice points that all represent infinite collections of the notional atoms we can start over the process of building compact proof paths. The elements of $\mathbf{2}^{\omega}$ are all subsets of ω. The quotient algebra $\frac{\mathbf{2}^{\omega}}{\mathbf{2}^{<\omega}}$ partitions $\mathbf{2}^{\omega}$ into segments each of which is isomorphic to $\mathbf{2}^{<\omega}$. A dilation of the boundary would map some collection of these $\mathbf{2}^{\omega}$ into the space originally occupied by $F(\omega) \cong \mathbf{2}^{<\omega}$.

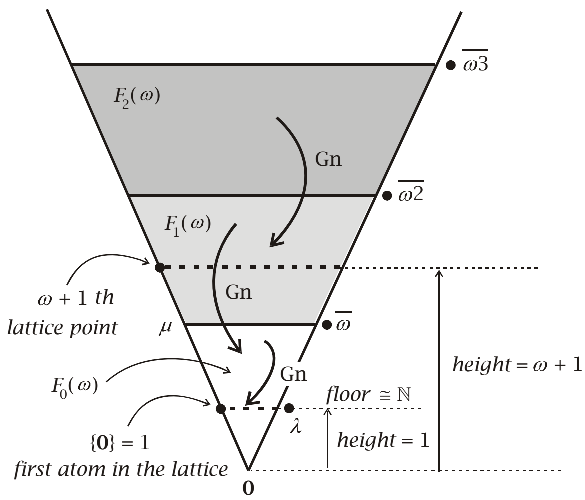
The atom λ is excluded from the filter $F(\omega) \cong \mathbf{2}^{<\omega}$. If it were included the filter would comprise the entire Cantor set, $\mathbf{2}^{\omega}$. The Cantor set comprises all subsets of ω. Since the ordinal $\omega+1 \not\subset \omega$, $\omega+1 \notin \mathbf{2}^{\omega}$. If we assume the Axiom of Choice, then the whole of $\mathbf{2}^{\omega}$ can be well ordered, so there is an $\omega+1$th lattice point. After ω the transfinite ordinals continue: -

$$\omega+1,\ \omega+2, \ldots, \omega 2,\ \omega 2+1,\ \omega 2+2,\ \ldots, \omega 3,\ \omega 3+1,\ \ldots,\ \omega 4,\ \ldots,\ \omega 5,\ \ldots,$$
$$\omega^2,\ \omega^2+1,\ \ldots,\ \omega^3,\ \omega^3+1,\ \ldots, \omega^4,\ \ldots, \omega^{\omega},\ \ldots, \omega^{\omega},\ \ldots, \omega^{\omega^{\omega}}, \ldots$$

These may act as *names* of the lattice points. When we do so we use the convention of an overbar to indicate this; for example, $\overline{\omega+1}$ indicates the *name* of the $(\omega+1)$th lattice point. We need a countable basis for the next filter up, and this is provided by $\overline{\omega+1}, \overline{\omega+2}, \ldots$; but $\overline{\omega+\omega} = \overline{\omega 2}$, must be excluded from this basis, for otherwise the basis becomes actually infinite and the lattice generated over it becomes uncountably infinite $\left(\cong 2^{\aleph_0}\right)$. So $\overline{\omega 2}$ must be the name of a set that is a member of the atom λ and the resultant is a sub-lattice, $F_1(\omega)$, isomorphic to $F_0(\omega) = F(\omega)$ that can be mapped into $F_0(\omega)$ by an extended Gödel map. Likewise, the next filter up, $F_2(\omega)$, will be the one with basis $\overline{\omega 2+1}, \overline{\omega 2+2}, \ldots$ and the extended Gödel maps moves this filter into the one with basis $\overline{\omega+1}, \overline{\omega+2}, \ldots$; the extended Gödel map is a systematic "shunt" of the partitions of the boundary towards the floor. By symmetry it also systematically shunts partitions of the boundary that are closer to the ceiling (in terms of the metric) towards the ceiling. It maps: -

$$\text{Gn}\begin{cases} F_0 \to \mathbb{N}^* & C_0 \to \mathbb{N}^{**} \\ F_{n+1} \to F_n & C_{n+1} \to C_n \end{cases}$$

The sets denoted $\mathbb{N}^*$ and $\mathbb{N}^{**}$ have to be clarified. The map $F_0 \to \mathbb{N}^*$ is onto the notional atoms of $\mathbf{2}^{<\omega}$ and these are $\mathbb{N}^* = \{\{0\},\{1\},\{2\},\ldots\}$ and not $\mathbb{N} = \{0,1,2,3,\ldots\}$. Similarly, we have $\mathbb{N}^{**} = \{\omega-\{0\}, \omega-\{1\}, \omega-\{2\}, \ldots\}$.



The length of a proof path from a lattice point to another lying it its filter shall be defined to be the minimum length of the path, where length is measured by the metric [Chap. 4 Sec. 5] of the lattice.[8] The maximum length of a proof path to a lattice point is its height above **0**. The diagram shows the filter $F_0(\omega)\vDash$. Informally we may describe this extended Gödel map as a "contraction" of the region $CF(\omega)\cong \mathbf{2}^{<\omega}$ onto $\mathbb{N}$ and a "dilation" of the boundary.

6.5 Definition, contraction

Let $f: M \to M$ be a map on a metric space (M,d) onto itself. Then f is a *contraction* if there exists a constant $K<1$ such that $d\big(f(x),f(y)\big)=K\,d(x,y)$ for all $x,y\in M$.

Formally, Gn as defined above, is *not* a contraction mapping, because the distance between lattice points is not always contracted. Therefore, we cannot apply the Banach fixed point theorem to this mapping: -

6.6 Banach fixed point theorem

Let $f: M \to M$ be a contraction of a complete metric space *M*. Then *f* has a unique fixed point $x\in M$.

This is to be expected, since the fixed points are not unique. The Cantor set is a complete metric space, but we can see that Gn has more than one fixed point. Under the specific definition for Gn given above $\mathrm{Gn}(\{0\})=\{0\}$, making $\{0\}$ into a fixed point. However, we shall show below [See 6.7 below] that λ is a fixed point.

Lattice points in $F_0(\omega)$ are finite lattice points that are numbered in some increasing sequence. For example, the lattice point $\{2,3,4\}$ corresponds to the number 9, in our sequence. The numeral $\overline{9}$ is a name of the lattice point $\{2,3,4\}$. Another representation of

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[8] The metric on $\mathbf{2}^{\omega}$ is the natural one defined in terms of height [4.5.8]. If $p,q\in\mathbf{2}^{\omega}$ then $d(p,q)=h(p)-h(y)$. With each lattice point x there is associated an ordinal β, which is its order isomorphism type – the ordinal to which it is similar. The ordinal to which the maximum of the lattice, **1**, corresponds is at this time unknown, since its determination is equivalent to a solution to the continuum hypothesis. The height of a lattice point $p\in\mathbf{2}^{\omega}$ is its ordinal, which is its height above the zero, **0**, of the lattice. The dual lattice where meets and joins are interchanged is an exact copy of $\mathbf{2}^{\omega}$ - only inverted. The map from the lattice to the dual is a reflection in the boundary, mapping all of $F(\omega)$ to $C(\omega)$ and, given the Boolean Prime Ideal theorem, $\mu\to\lambda=\mu'$. Hence $\mathbf{2}^{\omega}$ is subdivided into two regions M, the maximal ideal of $F(\omega)$ and Λ, the maximal filter of $C(\omega)$. The ordinal height above **0** of any element in $p\in\Lambda$ is greater than the ordinal height of $\mathbf{1}-p$ above **0**.

lattice points uses ascending primes; thus $\{2,3,4\}$ could be represented by $2\cdot 3\cdot 5=30$, and named by $\overline{30}$. A logic that is sufficiently strong permits prime factorisation, and hence implicitly encodes the natural numbers as a chain, as opposed to an anti-chain of partitions; it is this that permits Gödel numbering to be defined. Because of the presence of a chain and anti-chain defined over the same base set $\mathbb{N}=\{0,1,2,3,\ldots\}$ the underlying lattice is forced to adopt a complex structure that is equivalent to that of the Cantor set, $\mathbf{2}^{\omega}$. This structure is partitioned into analytic disks, each isomorphic to $F(\omega)\cong\mathbf{2}^{<\omega}$ upon which has been overlaid the possibility of a synthetic movement by complete induction that attains its boundary with is complement. Compact proof paths in the same analytic disk never attain the boundary, for that is the same as recursively reaching the last natural number, which is synthetically known to be impossible, though not analytically.

The lattice point named by $\overline{\omega}$ cannot be moved by the extended Godel map, Gn. It belongs to λ. The Gödel function cannot be defined for $\overline{\omega}$; this is because ω cannot be factorised into prime factors. For the same reasons all multiples $\overline{\omega}, \overline{\omega 2}, \overline{\omega 3}, \ldots, \overline{\omega^2}, \ldots, \overline{\omega^{\omega}}, \ldots, \overline{\omega^{\omega^{\omega\cdots}}}, \ldots$ denote lattice points belonging to λ. Hence all lattice points whose names are the additive, multiplicative, and exponential Hauptzahlen belong to λ.[9] In that part of the filter $F(\omega)\vDash$ that lies above $\overline{\omega}$, $\overline{\omega}$ takes on the role of the **0**, relative to all compact paths whose height is $>\omega$.

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[9] The operations of addition, multiplication and exponentiation are associated with certain numbers called *Hauptzahlen* that characterize them. (1) For addition, the Hauptzahlen are those numbers $\gamma>0$ such that $\gamma=\alpha+\gamma$ for all $0\le\alpha<\gamma$. Denote the set of all Haupzahlen by $\{\pi\}$; i.e. $\pi\in\{\pi\}$ iff $\alpha+\pi=\pi$. The smallest such π is 1, since $1=0+1$. It is the only finite π. The next is ω since for all $\nu<\omega$ we have $\omega=\nu+\omega$. Each Haupzahlen is irreducible and is therefore an *additive prime number*. (2) For transfinite ordinal multiplication let $\gamma>1$ be such that $\gamma=\alpha\cdot\gamma$ for all α such that $1\le\alpha<\gamma$. Then $2=1\cdot 2$ so 2 is a multiplicative Hauptzahlen and the only finite such. The multiplicative Hauptzhalen are only a part of the irreducible numbers, and only a small number of these are actually prime numbers. Of these that are both irreducible numbers the finite primes and ω are examples. Transfinite multiplicative Hauptzahlen are called δ-numbers. Each δ-number is also an additive Hauptzahlen. The smallest transfinite multiplicative Hauptzahlen is ω. Every limit number with exactly two factors and the irreducible limit numbers are also δ-numbers. Other δ-numbers exist of the form $\pi+1$ where π is an additive Hauptzahl. (3) For exponentiation, the ε-numbers are the Hauptzahlen of exponentiation, which are solutions to the equation $\omega^{\xi}=\xi$. $\omega^{\varepsilon_0}=\lim_{\nu}\{\omega,\omega^{\omega},\omega^{\omega^{\omega}},\ldots\}=\varepsilon_0$ is the least of all such ε-numbers. The sequence $\{\varepsilon_\nu\}$ is a well-ordered set of type Ω and hence of power $\aleph_1$. $\lim_{\nu}\{\varepsilon_\nu\}$ is also an ε-number. Every ε-number greater than ω is the limit of a sequence of multiplicative Hauptzahlen. The necessary and sufficient condition for determining the ε-numbers is the equation $\varepsilon=2^{\varepsilon}$.

6.7 Arithmetical hierarchy

Another version of the extended Gödel map can be defined without recourse to the Axiom of Choice. This is by means of the hierarchy of definable sets [See Arithmetic hierarchy – Chap.2 / 2.4.2]. Let $R_1, R_2, \ldots$ be an enumeration of all recursive relations of two variables: $R = R(x,y)$. The domain of each is $\omega \times \omega$. Then let an actually infinite set be defined by $(\exists x) R(x,y)$. To give a concrete example, let $R(x,y) \equiv (x = 2y)$; then $(\exists x) R(x,y) = \{0,2,4,\ldots\} = [2]$, where this is an actually infinite set, not co-finite, and hence belongs to the boundary. Such sets are said to be Σ_1^0. As they can be recursively enumerated, there is a countable basis of them that can be used as a skeleton to generate a filter. The dual to this filter is the ideal $(\forall x) R(x,y)$, which are denoted Π_1^0. The next filter up from $\text{filter}(\Sigma_1^0)$ is the $\text{filter}(\Sigma_2^0)$ comprising subsets of ω corresponding to definitions of the form $(\exists x)(\forall y) R(x,y,z)$ defined on the domain $\omega \times \omega \times \omega$. By iteration of this process we obtain a partition of the filter $F(\omega)\vDash$, and dually for the ideal defined on $C(\omega)$. The extended Gödel map, by numbering the lattice points in each filter, maps Σ_1^0 to $\mathbb{N}$, and in general Σ_{n+1}^0 to Σ_n^0. Unlike the previous partition based on the Axiom of Choice, this partition does not cover the whole of the boundary, because not every set is definable in this sense. However, the partition is infinite and hence the extended Gödel map defined upon it is inexhaustible. In this case, the class of indefinable sets is never moved by the extended Gödel map.

6.8 Fixed points

The lattice is $\mathbf{2}^\omega$ and it is partitioned into the finitely generated part of countable unions of subsets of $\mathbf{2}^\omega$ and denoted $\mathbf{2}^{<\omega}$ and its complement is $\mathbf{2}^\omega - \mathbf{2}^{<\omega}$. We have: -

$$\text{Gn}\begin{cases} F_0 \to \mathbb{N}^* & C_0 \to \mathbb{N}^{**} \\ F_{n+1} \to F_n & C_{n+1} \to C_n \end{cases}$$

Every lattice point in $CF_0(\omega)$ is the name of a compact proof, $\Gamma \vdash X$, path in $\mathbf{2}^\omega$ and corresponds to a Gödel number, $\text{Pf}(\ulcorner\Gamma\urcorner, \ulcorner X\urcorner)$. Hence: -

$$\text{Gn: } (\Gamma \vdash X) \in CF(\omega) \text{ iff } \{\text{Pf}(\ulcorner\Gamma\urcorner, \ulcorner X\urcorner)\}$$

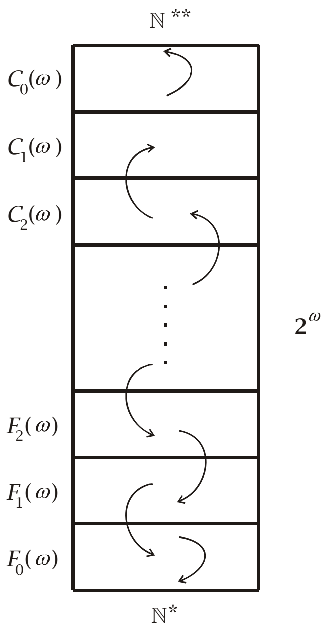
The filter $F_0(\omega) \equiv F(\omega)\vdash$ is given by $\{\phi \in \mathbf{2}^\omega : (\exists \alpha \in A)\alpha \vdash \phi\}$ were A is the set of atoms of 2^ω - that is, it is the set of all finite lattice points that can be reached by compact proof paths from finite atoms of $\mathbf{2}^{\omega}$. The lattice points in $C_0(\omega)$ also denote finite proof paths. The complement of $(CF)_0(\omega) \equiv CF(\omega)\vdash$ is $\mathbf{2}^{\omega} - (CF)_0 \equiv CF(\omega)\nvdash$ - it is the set of all lattice points that do not denote compact proof paths from any atoms. This may be denoted $\neg\mathrm{Pf}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)$. This states: -

$\neg\mathrm{Pf}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)$ iff Γ is not a compact proof of X
iff $X \notin \mathrm{filter}(\Gamma)$

There are two ways in which we can have $\neg\mathrm{Pf}(\ulcorner\Gamma\urcorner,\ulcorner X\urcorner)$: -

1. There can be a proof of $\ulcorner X\urcorner$, only it is not compact, and
2. There is no proof whatsoever of $\ulcorner X\urcorner$.

Now Gn maps $(CF)_0(\omega) \to \mathbb{N}^* \cup \mathbb{N}^{**}$ and shunts all the other filters $(CF)_\beta$ where β is an ordinal.



The atom $\lambda = \{\mathbb{N}\}$ could not be moved by Gn. This is because $\mathbb{N}$ cannot be factorised. Therefore, it is a fixed point of Gn. Furthermore, if this atom is ever was moved into $\mathbf{2}^{<\omega}$ after k iterations of Gn, then we have a proof of everything that lies in the filter generated by λ. That would amount to a proof of every formula corresponding to an infinite set whatsoever, so would entail the inconsistency of all mathematics. This shows that *if mathematics is consistent* then we cannot move the fixed point. As **1** lies in $\mathrm{Fil}(\lambda)$ then mapping λ into $\mathbf{2}^{<\omega}$

would provide a compact proof of what was originally an uncountably long proof of the inconsistency of mathematics. The same applies to μ. If we ever move either μ or λ into the compact region as a result of a mapping *f* then we shall have mapped the infinite into the finite and therefore have exhausted the infinite. This will result in inconsistency. Thus the embedding of arithmetic within an analytic formal logic is consistent only if μ and λ are fixed points of Gn.

6.9 Interpretation of the Gödel sentence

Let $\mathrm{K}_0 \equiv \{0\} \wedge \{1\} \wedge \{2\} \wedge \ldots$ be a sufficiently strong theory true of all atoms in M. Let Q_0 be the Gödel-Rosser sentence relative to K_0. Then $Q_0 = \mathrm{K}_0 \nvdash Q_0$ and $\mathrm{K}_0 \vDash Q_0$. Q_0 is not a lattice point that can be reached by any compact proof path in M - this is what $Q_0 = \mathrm{K}_0 \nvdash Q_0$ means. But $\mathrm{K}_0 \equiv \{0\} \wedge \{1\} \wedge \{2\} \wedge \ldots$ represents all the atoms of the partition whatsoever <u>except</u> λ; so what else is there to infer up the lattice from K_0 except λ? There is nothing else that can be added to be inferred. Hence $Q_0 \equiv \lambda$. But $\lambda = \{\mathbb{N}\}$ is the atom that adds the set $\mathbb{N}$ as a chain equipped with complete induction to the lattice. Hence we may interpret Q_0 as a well-ordered set with complete induction. Gödel's theorem, $\mathrm{K}_0 \vDash \lambda_0$ but $\mathrm{K}_0 \nvdash \lambda_0$, may be interpreted as *saying* complete induction cannot be inferred from a formal analytic logic, and one cannot (in the absence of the Axiom of Choice) obtain the chain $\mathbb{N}$ from the antichain μ that is equinumerous to it. Gödel's theorem is true precisely because Poincaré's thesis is true. The underlying facts on which it is built is this: *a chain is not an antichain* and the potential infinite is not actually infinite.

We can form a succession of versions of the chain $\mathbb{N}$: $\lambda_0, \lambda_1, \lambda_2, \ldots$ and import these into the lattice. For example, when we form first-order Peano arithmetic [Chap.2 / 2.12] we import a potentially countably infinite <u>but actually finite</u> number of these into the compact part of the lattice, adding them there as <u>particular instances</u> of the chain $\mathbb{N}$ and its associated principle of complete induction. Gödel's theorem shows us that we could never exhaust the potential of complete induction by such a process – for there will always be a λ in any lattice corresponding to a sufficiently strong logic. Thus, Poincaré's thesis is upheld.

We also see why there are two versions of Gödel's theorem – two Gödel sentences – the one that permits violation of ω-consistency [Sec. 1 above] and the other that does not [Sec. 2]. We saw in the chapter on generic sets that generic sequences define transcendental numbers that are elements of non-standard models. Therefore, there is a version of Gödel's theorem that permits this, and there is a version that does not. It is the Gödel-Rosser sentence $Q' \equiv \lambda$ that disallows non-standard models; the Gödel sentence, *Q*, permits any member of the boundary as an element of a non-standard model of arithmetic that demonstrates the incompleteness of first-order arithmetic. In first-order arithmetic we can create generic sequences, and hence we cannot rule out non-standard models. That is why first-order arithmetic fails to be categorical.